SELECTED PROBLEMS OF UNCERTAINTY ESTIMATES FOR NON-LINEAR DISCRETE-TIME SYSTEMS

Przemysław Orłowski

Technical University of Szczecin, Control Engineering Institute,
Sikorskiego 37, 70-313 Szczecin, e-mail: orzel@we.ps.pl

1. INTRODUCTION

A lot of real control systems are non-linear and/or time-varying. The most general models for control system are given in state space. For synthesis robust control systems very often we need to estimate the maximal differences of the output signals. In this paper the most generalized case of state space model, with non-linear and time-varying coefficients will be presented.

Troughout the paper it is assumed, that a positive integer N and a *nominal control* $\mathbf{u}_p(\cdot)\in(\mathbf{R}^m)^N$ which has feedback form $\mathbf{u}_p(\cdot)=\mathbf{v}_p(\cdot)+\mathbf{F}(\cdot)\cdot\mathbf{x}_p(\cdot)$ where $\mathbf{v}_p(\cdot)\in(\mathbf{R}^m)^N$, $\mathbf{F}(\cdot)\in L((\mathbf{R}^p,\mathbf{R}^m)^N)$, N is a fixed step are given. The corresponding *nominal state* and *nominal output* functions of Σ are denoted by $\mathbf{x}_p(\cdot)\in(\mathbf{R}^n)^N$ and $\mathbf{y}_p(\cdot)\in(\mathbf{R}^p)^N$. If this control will be applied this control to the uncertain system then, in general $\mathbf{y}_\Delta(N)\neq\mathbf{y}_p(N)$ and $\mathbf{y}_\Delta(\cdot)\neq\mathbf{y}_p(\cdot)$.

The main purpose of this paper is to develop techniques for estimating the difference $\left\|\mathbf{x}_\Delta(\cdot)-\mathbf{x}_p(\cdot)\right\|_{(\mathbf{R}^p)^N}$ in term of the bounds δ_A, δ_B, δ_C, δ_{Ar}, δ_{Br}, δ_{Cr} and nominal system parameters $\mathbf{A}(\mathbf{x}_p(k),k)$, $\mathbf{B}(\mathbf{u}_p(k),k)$ and $\mathbf{C}(\mathbf{x}_p(k),k)$. for uncertain, linear, non-stationary discrete-time systems on a finite time horizon. Uncertainty in the system description is modelled by an unknown (norm bounded) additive perturbation of the system matrix, dependent on the state and input.

2. THE METRIC SPACE

Space of vector's sequence are given by Hilbert space (l^2).

$$(\mathbf{R}^q)^N=\mathbf{R}^q\times\mathbf{R}^q\times\ldots\times\mathbf{R}^q \tag{1}$$

Elements of the space are sequences of vectors

$$\mathbf{z}=[\mathbf{z}(0)\ldots\mathbf{z}(N-1)]^{\mathrm{T}} \tag{2}$$

where $\mathbf{z}(i)\in\mathbf{R}^q$.

Scalar products in $(\mathbf{R}^q)^N$ is defined as follow

$$\langle\mathbf{z},\mathbf{v}\rangle_{(\mathbf{R}^q)^N}=\sum_{i=0}^{N-1}\langle\mathbf{z}(i),\mathbf{v}(i)\rangle_{\mathbf{R}^q}=\sum_{i=0}^{N-1}\mathbf{z}^{\mathrm{T}}(i)\cdot\mathbf{v}(i) \tag{3}$$

where $\mathbf{z},\mathbf{v}\in(\mathbf{R}^q)^N$.

The induced norm has the form

$$\left\|\mathbf{z}\right\|^2_{(\mathbf{R}^q)^N}=\sum_{i=0}^{N-1}\langle\mathbf{z}(i),\mathbf{z}(i)\rangle_{\mathbf{R}^q}=\sum_{i=0}^{N-1}\mathbf{z}^{\mathrm{T}}(i)\cdot\mathbf{z}(i) \tag{4}$$

where $\mathbf{z}\in(\mathbf{R}^q)^N$.

Space of matrix's sequence are given by Hilbert space.

$$(\mathbf{R}^p,\mathbf{R}^q)^N=(\mathbf{R}^p\times\mathbf{R}^q)\times(\mathbf{R}^p\times\mathbf{R}^q)\times\ldots\times(\mathbf{R}^p\times\mathbf{R}^q) \tag{5}$$

Elements of the space are sequences of matrices

$$\mathbf{Z}=[\mathbf{Z}(0)\ldots\mathbf{Z}(N-1)]^{\mathrm{T}} \tag{6}$$

where $\mathbf{Z}(i) \in (\mathbf{R}^p, \mathbf{R}^q)$.

The norm has form

$$\|\mathbf{Z}\|^2_{(\mathbf{R}^p,\mathbf{R}^q)^N} = \max_{j,k} \lambda_{j,k} \qquad (7)$$

where $\lambda_{j,k}$ are eigenvalues of $\mathbf{Z}^{\mathrm{T}}(j) \cdot \mathbf{Z}(j)$, k=1,2,...,q j=1,2,...,N.

Norm of vector's or matrix's space transformation is defined as follow

$$\|\mathbf{M}^{\mathrm{F}}\| = \sup_{\mathbf{h}\in(\mathbf{R}^q)^N} \frac{\|\mathbf{M}^{\mathrm{F}}\mathbf{h}\|}{\|\mathbf{h}\|} \qquad (8)$$

3. NOMINAL CONTROL SYSTEM

The *nominal*, unperturbed control system Σ has the form

$$\mathbf{x}_p(k+1) = \mathbf{A}(\mathbf{x}_p(k),k) \cdot \mathbf{x}_p(k) + \mathbf{B}(\mathbf{u}_p(k),k) \cdot \mathbf{u}_p(k)$$

$$\mathbf{x}_p(0)=\mathbf{x}_0, \qquad (9)$$

$$\mathbf{y}_p(k) = \mathbf{C}(\mathbf{x}_p(k),k) \cdot \mathbf{x}_p(k) \qquad k\in\mathbf{N}, \qquad (10)$$

where $\mathbf{x}_p(\cdot) \in (\mathbf{R}^n)^N$ is the nominal state, $\mathbf{u}_p(\cdot) \in (\mathbf{R}^m)^N$ the nominal control, $\mathbf{y}_p(\cdot) \in (\mathbf{R}^p)^N$ is the nominal output, and $\mathbf{A}(\mathbf{x}_p(k),k) \in L(\mathbf{R}^n)$, $\mathbf{B}(\mathbf{u}_p(k),k) \in L(\mathbf{R}^m, \mathbf{R}^n)$, $\mathbf{C}(\mathbf{x}_p(k),k) \in L(\mathbf{R}^n, \mathbf{R}^p)$ are known matrices' functions.

In order to cover the most general situation we assume that control has the following feedback form

$$\mathbf{u}_p(k) = \mathbf{v}_p(k) + \mathbf{F}(k) \cdot \mathbf{x}_p(k) \qquad (11)$$

where $\mathbf{v}_p(\cdot) \in (\mathbf{R}^m)^N$, $\mathbf{F}(\cdot) \in L((\mathbf{R}^p, \mathbf{R}^m)^N)$ and k=0,1,...,N-1.

Substituting (11) into (9-10) state equations will have the form

$$\mathbf{x}_p(k+1) = (\mathbf{A}(\mathbf{x}_p(k),k) + \mathbf{B}(\mathbf{u}_p(k),k) \cdot \mathbf{F}(k)) \cdot \mathbf{x}_p(k)$$
$$+\mathbf{B}(\mathbf{u}_p(k),k) \cdot \mathbf{v}_p(k) \qquad (12)$$

$$\mathbf{y}_p(k) = \mathbf{C}(\mathbf{x}_p(k),k) \cdot \mathbf{x}_p(k),$$

$$\mathbf{x}_p(0)=\mathbf{x}_0, \quad k=0,1,...,N-1, \qquad (13)$$

4. PERTURBED CONTROL SYSTEM

Real control system is different from (9-10) and may be described by *perturbed* model Σ_Δ as follow

$$\mathbf{x}_\Delta(k+1) = \mathbf{A}_\Delta(\mathbf{x}_\Delta(k),k) \cdot \mathbf{x}_\Delta(k) + \mathbf{B}_\Delta(\mathbf{u}_\Delta(k),k) \cdot \mathbf{u}_\Delta(k)$$

$$\mathbf{x}_\Delta(0)=\mathbf{x}_0, \qquad (14)$$

$$\mathbf{y}_\Delta(k) = \mathbf{C}_\Delta(\mathbf{x}_\Delta(k),k) \cdot \mathbf{x}_\Delta(k) \qquad k\in\mathbf{N}, \qquad (15)$$

The control has the same form as for the *nominal* system.

It has been assumed, that the state uncertainty could be described by following model.

For matrix **A**

$$\mathbf{A}_\Delta(\mathbf{x}_\Delta(k),k) \cdot \mathbf{x}_\Delta(k) = \mathbf{A}(\mathbf{x}_p(k),k) \cdot \mathbf{x}_\Delta(k) +$$
$$+\Delta_{\mathrm{A}}(\mathbf{x}_p(k),k) \cdot \mathbf{x}_\Delta(k) + \Delta'_{\mathrm{Ar}}(\mathbf{x}_p(k),k) \cdot (\mathbf{x}_\Delta(k) - \mathbf{x}_p(k))$$
$$(16)$$

where $\Delta_{\mathbf{A}}(\mathbf{x}_p(k),k) \in L(\mathbf{R}^n, \mathbf{R}^n)$, $\Delta'_{\mathbf{Ar}}(\mathbf{x}_p(k),k) \in L(\mathbf{R}^n, \mathbf{R}^n)$, k=0,1,...,N-1, and

$$\|\Delta_{\mathbf{A}}(\mathbf{x}_p(k),k)\| \le \delta_A < \infty, \qquad (17)$$

$$\|\Delta'_{\mathbf{Ar}}(\mathbf{x}_p(k),k)\| \le \delta_{Ar} < \infty, \qquad (18)$$

For matrix **B**

$$\mathbf{B}_\Delta(\mathbf{u}_\Delta(k),k) \cdot \mathbf{u}_\Delta(k) = \mathbf{B}(\mathbf{u}_p(k),k) \cdot \mathbf{u}_\Delta(k) +$$
$$+\Delta_{\mathrm{B}}(\mathbf{u}_p(k),k) \cdot \mathbf{u}_\Delta(k) + \Delta'_{\mathrm{Br}}(\mathbf{u}_p(k),k) \cdot (\mathbf{u}_\Delta(k) - \mathbf{u}_p(k))$$
$$(19)$$

where $\Delta_{\mathbf{B}}(\mathbf{u}_p(k),k) \in L(\mathbf{R}^n, \mathbf{R}^n)$, $\Delta'_{\mathbf{Br}}(\mathbf{u}_p(k),k) \in L(\mathbf{R}^n, \mathbf{R}^n)$, k=0,1,...,N-1, and

$$\|\Delta_{\mathbf{B}}(\mathbf{u}_p(k),k)\| \le \delta_B < \infty, \qquad (20)$$

$$\|\Delta'_{\mathbf{Br}}(\mathbf{u}_p(k),k)\| \le \delta_{Br} < \infty, \qquad (21)$$

For matrix **C**

$$\mathbf{C}_\Delta(\mathbf{x}_\Delta(k),k) \cdot \mathbf{x}_\Delta(k) = \mathbf{C}(\mathbf{x}_p(k),k) \cdot \mathbf{x}_\Delta(k) +$$
$$+\Delta_{\mathrm{A}}(\mathbf{x}_p(k),k) \cdot \mathbf{x}_\Delta(k) + \Delta'_{\mathrm{Ar}}(\mathbf{x}_p(k),k) \cdot (\mathbf{x}_\Delta(k) - \mathbf{x}_p(k))$$
$$(22)$$

where $\Delta_{\mathrm{C}}(\mathbf{x}_p(k),k) \in L(\mathbf{R}^p, \mathbf{R}^n)$, $\Delta'_{\mathbf{Cr}}(\mathbf{x}_p(k),k) \in L(\mathbf{R}^p, \mathbf{R}^n)$, k=0,1,...,N-1, and

$$\|\Delta_{\mathrm{C}}(\mathbf{x}_p(k),k)\| \le \delta_C < \infty, \qquad (23)$$

$$\|\Delta'_{\mathbf{Cr}}(\mathbf{x}_p(k),k)\| \le \delta_{Cr} < \infty, \qquad (24)$$

To obtain the norm of maximal output deviation, we needn't know the uncertainty matrices $\mathbf{\Delta_A}$, $\mathbf{\Delta_B}$, $\mathbf{\Delta_C}$, $\mathbf{\Delta'_{Ar}}$, $\mathbf{\Delta'_{Br}}$, $\mathbf{\Delta'_{Cr}}$, we have to know only their estimates δ_A, δ_B, δ_C, δ_{Ar}, δ_{Br}, δ_{Cr}.

5. OUTPUT UNCERTAINTY ESTIMATION

For the sake of simplicity we introduce two operators $\mathbf{L^F} \in L((\mathbf{R}^n)^N, (\mathbf{R}^n)^N)$ and $\mathbf{N^F} \in L(\mathbf{R^n}, (\mathbf{R}^n)^N)$, defined as follows

$$
\begin{aligned}
(\mathbf{L^F}(\mathbf{B}\cdot\mathbf{v}_p))(k) = \sum_{i=0}^{k-2}\left[\prod_{j=i+1}^{k-1}\begin{pmatrix}\mathbf{A}(\mathbf{x}_p(j), j) + \\ \mathbf{B}(\mathbf{v}_p(j) + \mathbf{x}_p(j)\cdot\mathbf{F}(j), j)\cdot\mathbf{F}(j)\end{pmatrix}\right] \\
\cdot\mathbf{B}(\mathbf{v}_p(i) + \mathbf{x}_p(i)\cdot\mathbf{F}(i), i)\cdot\mathbf{v}_p(i) \\
+\mathbf{B}(\mathbf{v}_p(k-1) + \mathbf{x}_p(k-1)\cdot\mathbf{F}(k-1), k-1)\cdot\mathbf{v}_p(k-1)
\end{aligned} \tag{25}
$$

$$
(\mathbf{N^F}\mathbf{x}_0)(k) = \prod_{j=0}^{k-1}\begin{pmatrix}\mathbf{A}(\mathbf{x}_p(j), j) + \\ \mathbf{B}(\mathbf{v}_p(j) + \mathbf{x}_p(j)\cdot\mathbf{F}(j), j)\cdot\mathbf{F}(j)\end{pmatrix}\cdot\mathbf{x}_0 \tag{26}
$$

where k=2,3,...,N.

Theorem 1. For every system Σ_p described by equations (9-10) the state and output trajectory can be written as follows

$$
\mathbf{x}_p(k) = (\mathbf{N^F}\mathbf{x}_0)(k) + (\mathbf{L^F}(\mathbf{B}\cdot\mathbf{v}_p))(k) \tag{27}
$$

$$
\mathbf{y}_p(k) = \mathbf{C}(\mathbf{x}_p(k), k)\cdot\mathbf{x}_p(k)\,, \tag{28}
$$

Theorem 2. For every perturbed system Σ_Δ described by equations (14-15) the state and output trajectory can be written as follows

$$
\begin{aligned}
\mathbf{x}_\Delta(k) &= \mathbf{x}_p(k) + \mathbf{L^F}(\Delta_A(\mathbf{x}_p(k), k)\cdot\mathbf{x}_\Delta(k))(k) \\
&+ \mathbf{L^F}(\Delta'_{Ar}(\mathbf{x}_p(k), k)\cdot(\mathbf{x}_\Delta(k) - \mathbf{x}_p(k)))(k) \\
&+ \mathbf{L^F}(\Delta_B(\mathbf{v}_p(k) + \mathbf{F}(k)\cdot\mathbf{x}_p(k))\cdot(\mathbf{v}_\Delta(k) + \mathbf{F}(k)\cdot\mathbf{x}_\Delta(k))(k) \\
&+ \mathbf{L^F}(\Delta'_{Br}(\mathbf{v}_p(k) + \mathbf{F}(k)\cdot\mathbf{x}_p(k)) \\
&\cdot(\mathbf{v}_\Delta(k) - \mathbf{v}_p(k) + \mathbf{F}(k)\cdot\mathbf{x}_\Delta(k) - \mathbf{F}(k)\cdot\mathbf{x}_p(k))(k)
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\mathbf{y}_\Delta(k) = \mathbf{C}(\mathbf{x}_p(k), k)\cdot\mathbf{x}_\Delta(k) + \Delta_C(\mathbf{x}_p(k), k)\cdot\mathbf{x}_\Delta(k) \\
+ \Delta'_{Cr}(\mathbf{x}_p(k), k)\cdot(\mathbf{x}_\Delta(k) - \mathbf{x}_p(k))
\end{aligned} \tag{30}
$$

Bellow we derive explicit expressions for the difference $\|\mathbf{x}_\Delta(\cdot) - \mathbf{x}_p(\cdot)\|_{(\mathbf{R}^p)^N}$.

6. TRAJECTORY DEVIATION NORM

Theorem 3. *For every* $\Delta_\mathbf{A} \in L(\mathbf{R}^n, \mathbf{R}^n)^N$, $\Delta_\mathbf{B} \in L(\mathbf{R}^n, \mathbf{R}^m)^N$, $\Delta_\mathbf{C} \in L(\mathbf{R}^p, \mathbf{R}^n)^N$, $\Delta'_\mathbf{Ar} \in L(\mathbf{R}^n, \mathbf{R}^n)^N$, $\Delta'_\mathbf{Br} \in L(\mathbf{R}^n, \mathbf{R}^m)^N$, $\Delta'_{\mathbf{C}r} \in L(\mathbf{R}^p, \mathbf{R}^n)^N$ when equations (16-24) and

$$
\left(\delta_A + \delta_B\cdot\|\mathbf{F}\|_{(\mathbf{R}^m,\mathbf{R}^n)^N} + \delta_{Ar} + \delta_{Br}\cdot\|\mathbf{F}\|_{(\mathbf{R}^m,\mathbf{R}^n)^N}\right) < \left\|\mathbf{L^F}\right\|^{-1} \tag{31}
$$

are satisfied, the distance $\|\mathbf{x}_\Delta(\cdot) - \mathbf{x}_p(\cdot)\|_{(\mathbf{R}^p)^N}$ can be estimated as follows

$$
\|\mathbf{x}_\Delta(\cdot) - \mathbf{x}_p(\cdot)\|_{(\mathbf{R}^n)^N} \le \frac{\|\mathbf{L^F}\|\cdot(\delta_{aoz} + \delta_{Axz}\cdot\|\mathbf{x}_p\|)}{1 - \|\mathbf{L^F}\|\cdot(\delta_{Axz} + \delta'_{Arz})} \tag{32}
$$

where

$$
\delta_{Axz} = \delta_{Ax} + \delta_{Bu}\cdot\|\mathbf{F}\| \tag{33}
$$

$$
\delta'_{Arz} = \delta'_{Ar} + \delta'_{Br}\cdot\|\mathbf{F}\| \tag{34}
$$

$$
\delta_{aoz} = \delta_{Bu}\cdot\|\mathbf{v}_\Delta\| + \delta'_{Br}\cdot\|\mathbf{v}_\Delta - \mathbf{v}_p\| \tag{35}
$$

Proof: It is a standard result of functional analysis, if we transform (29) with triangle inequality and (16-23) are satisfying there is

$$
\begin{aligned}
\|\mathbf{x}_\Delta - \mathbf{x}_p\|_{(\mathbf{R}^n)^N} \le \|\mathbf{L^F}\|\cdot\delta_{Ax}\cdot\|\mathbf{x}_\Delta\| + \|\mathbf{L^F}\|\cdot\delta'_{Ar}\cdot\|\mathbf{x}_\Delta - \mathbf{x}_p\| \\
+\|\mathbf{L^F}\|\cdot\delta_{Bu}\cdot\left[\|\mathbf{v}_\Delta\| + \|\mathbf{F}\|\cdot\|\mathbf{x}_\Delta\|\right] \\
+\|\mathbf{L^F}\|\cdot\delta'_{Br}\cdot\left[\|\mathbf{v}_\Delta - \mathbf{v}_p\| + \|\mathbf{F}\|\cdot\|\mathbf{x}_\Delta - \mathbf{x}_p\|\right]
\end{aligned}
$$

then

$$
\|\mathbf{x}_\Delta - \mathbf{x}_p\|_{(\mathbf{R}^n)^N} \le \|\mathbf{L^F}\|\cdot\begin{bmatrix}\delta_{Axz}\cdot\|\mathbf{x}_\Delta\| + \delta_{aoz} \\ +\delta'_{Arz}\cdot\|\mathbf{x}_\Delta - \mathbf{x}_p\|\end{bmatrix}
$$

$$
\begin{aligned}
\|\mathbf{x}_\Delta - \mathbf{x}_p\|_{(\mathbf{R}^n)^N}\cdot\left[1 - \delta'_{Arz}\cdot\|\mathbf{L^F}\|\cdot\|\mathbf{x}_\Delta - \mathbf{x}_p\|\right] \le \\
\le \|\mathbf{L^F}\|\cdot\left[\delta_{Axz}\cdot\|\mathbf{x}_\Delta\| + \delta_{aoz}\right]
\end{aligned}
$$

$$
\|\mathbf{x}_\Delta - \mathbf{x}_p\|_{(\mathbf{R}^n)^N} \le \frac{\|\mathbf{L^F}\|\cdot\left[\delta_{Axz}\cdot\|\mathbf{x}_\Delta\| + \delta_{aoz}\right]}{1 - \delta'_{Arz}\cdot\|\mathbf{L^F}\|} \tag{36}
$$

$$
\begin{aligned}
\|\mathbf{x}_\Delta\|_{(\mathbf{R}^n)^N} \le \|\mathbf{x}_p\| + \\
+\|\mathbf{L^F}\|\cdot\left[\delta_{Axz}\cdot\|\mathbf{x}_\Delta\| + \delta'_{Arz}\cdot\|\mathbf{x}_\Delta - \mathbf{x}_p\| + \delta_{aoz}\right]
\end{aligned}
$$

$$
\|\mathbf{x}_\Delta\|_{(\mathbf{R}^n)^N} \le \frac{\|\mathbf{x}_p\| + \|\mathbf{L^F}\|\cdot\left[\delta'_{Arz}\cdot\|\mathbf{x}_\Delta - \mathbf{x}_p\| + \delta_{aoz}\right]}{1 - \|\mathbf{L^F}\|\cdot\delta_{Axz}} \tag{37}
$$

By substituting (37) into (36) we have the state trajectory estimate.

$$\left\|\mathbf{x}_\Delta - \mathbf{x}_p\right\|_{(\mathbf{R}^n)^N} \leq$$
$$\frac{\left\|\mathbf{L^F}\right\| \cdot \delta_{\mathrm{Axz}} \cdot \left[\left\|\mathbf{x}_p\right\| + \left\|\mathbf{L^F}\right\| \cdot \left[\delta'_{\mathrm{Arz}} \cdot \left\|\mathbf{x}_\Delta - \mathbf{x}_p\right\| + \delta_{\mathrm{aoz}}\right]\right]}{\left[1 - \delta'_{\mathrm{Arz}} \cdot \left\|\mathbf{L^F}\right\|\right] \cdot \left[1 - \left\|\mathbf{L^F}\right\| \cdot \delta_{\mathrm{Axz}}\right]}$$
$$+\frac{\left\|\mathbf{L^F}\right\| \cdot \delta_{\mathrm{aoz}}}{1 - \delta'_{\mathrm{Arz}} \cdot \left\|\mathbf{L^F}\right\|} \tag{38}$$

After simplifying (38), and when equations (31) and (16-24) are satisfied the norm of uncertain state's deviation we can write as follow

$$\left\|\mathbf{x}_\Delta - \mathbf{x}_p\right\|_{(\mathbf{R}^n)^N} \leq \frac{\left\|\mathbf{L^F}\right\| \cdot \left[\delta_{\mathrm{Axz}} \cdot \left\|\mathbf{x}_p\right\| + \delta_{\mathrm{aoz}}\right]}{1 - \left\|\mathbf{L^F}\right\| \cdot \left[\delta_{\mathrm{Axz}} + \delta'_{\mathrm{Arz}}\right]}$$

It is equivalent to equation (32). □

7. NORMS OF OPERATORS

It follows from the above formulas that effectivness of the estimate (32) will highly depend on how good are the estimates of the operator norms ||**C**·**L**$^{\mathbf{F}}$||, ||**L**$^{\mathbf{F}}$||. In this section, a method which allows to obtain a very tight estimates for these norms is presented. The main idea is based on the following general result.

Lemma 4 *Let U, Y be real Hilbert spaces, F∈L(U,Y), y_0∈Y, γ∈(0,∞) and J(u) be a functional defined on U and given by*

$$J(u) = \left\|Fu + y_0\right\|_Y^2 - \gamma^2 \cdot \left\|u\right\|_U^2 \tag{39}$$

(a) $\left\|F\right\| < \gamma$ if and only if there exists *β >0, such that*

$$\left\|Fu\right\|_Y^2 - \gamma^2 \cdot \left\|u\right\|_U^2 \leq -\beta \cdot \left\|u\right\|_U^2 \qquad \forall u \in U \tag{40}$$

Consequently, if $\left\|F\right\| < \gamma$*, then* (39) *always achieves a unique finite maksimum over U.*

(b) If $\left\|F\right\| > \gamma$ *then* (39) *does not achieve a finite maximum over U, i.e.* $\sup_{u \in U} J(u) = +\infty$.

It follows from this lemma that $\left\|F\right\| = \inf \gamma$ over all γ such that the maximization of (39) has a finite solution . The required value of γ can be found with arbitrary accuracy, e.g. by means of the bisection method. In our case the operator *F* is $\mathbf{C} \cdot \mathbf{L^F} \in L((\mathbf{R}^n)^N, (\mathbf{R}^n)^N)$. We can exploit the equivalence between the maximization of the functional (39) and the existence of a solution to the corresponding Riccati difference equations. Namely the following results hold.

Theorem 5. $\left\|\mathbf{C} \cdot \mathbf{L^F}\right\| < \gamma$ *if and only if the following difference Riccati equation*

$$\begin{aligned}
\mathbf{R}(k) = {} & (\mathbf{A} + \mathbf{B} \cdot \mathbf{F}(k))^{\mathrm{T}} \cdot \mathbf{R}(k+1) \cdot (\mathbf{A} + \mathbf{B} \cdot \mathbf{F}(k)) \\
& - (\mathbf{A} + \mathbf{B} \cdot \mathbf{F}(k))^{\mathrm{T}} \cdot \mathbf{R}(k+1) \cdot [\mathbf{R}(k+1) \\
& - \gamma^2 \cdot \mathbf{I}]^{-1} \cdot \mathbf{R}(k+1) \cdot (\mathbf{A} + \mathbf{B} \cdot \mathbf{F}(k)) + \mathbf{C}^{\mathrm{T}} \cdot \mathbf{C} \\
& \mathbf{R}(N) = 0
\end{aligned} \tag{41}$$

has a symmetric solution $\mathbf{R}(k) \in L(\mathbf{R}^{\mathbf{n}})$, *k=0,1,...,N-1.*

8. CONCLUSION

The derived estimates could be very useful in analysis and synthesis control systems. It is possible to develop other than those presented in this paper estimates for the deviations under consideration. However, it seems to be true that tight estimates for operator norms ||**C**·**L**$^{\mathbf{F}}$||, ||**L**$^{\mathbf{F}}$|| will always play a crucial role. For this reason, the presented method provides a very effective solution to this problem. The developed estimates can be used in various control desgn tasks for perturbed non-linear discrete time systems.

Abstract

The paper develops a mathematical framework which helps to analyse the following class of finite horizon control problems for uncertain nonlinear discrete-time systems.

$$\mathbf{x}_p(k+1) = \mathbf{A}(\mathbf{x}_p(k), k) \cdot \mathbf{x}_p(k) + \mathbf{B}(\mathbf{u}_p(k), k) \cdot \mathbf{u}_p(k)$$

$$\mathbf{y}_p(k) = \mathbf{C}(\mathbf{x}_p(k), k) \cdot \mathbf{x}_p(k),$$

$\mathbf{x}_p(0)=\mathbf{x}_0$, k=0,1,...,N-1,

where $\mathbf{x}_p(\cdot) \in (\mathbf{R}^n)^N$ is the nominal state, $\mathbf{u}_p(\cdot) \in (\mathbf{R}^m)^N$ the nominal control, $\mathbf{y}_p(\cdot) \in (\mathbf{R}^p)^N$ is the nominal output, and $\mathbf{A}(\mathbf{x}_p(k), k) \in L(\mathbf{R}^n)$, $\mathbf{B}(\mathbf{u}_p(k), k) \in L(\mathbf{R}^m, \mathbf{R}^n)$ and $\mathbf{C}(\mathbf{x}_p(k), k) \in L(\mathbf{R}^n, \mathbf{R}^p)$ are known matrices' functions.

Uncertainty in the system description are modelled by unknown (norm bounded) mixed additive-multiplicative perturbations of the system matrix.

It is assumed that the nominal control has feedback form. If the nominal control will be applied to the uncertain system the state and output will be (in general) different. Formulas for deriving estimates for the deviations of the output of perturbed system from the output of the nominal one has been presented. These estimates use norms of certain dynamical operators defined on a finite time interval.

Keywords: Uncertain dynamic systems, uncertain linear systems, discrete-time systems, error estimation, non-linear systems, non-linear analysis, time-varying systems, state-space models.

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M.Sc. Eng. Przemysław Orłowski
Technical University of Szczecin
Control Engineering Institute
Sikorskiego 37, 70-313 Szczecin
tel. (091)4495407, (091)4495411
e-mail: orzel@we.ps.pl