ROBUST CONTROL DESIGN FOR A CLASS OF NON-LINEAR SYSTEMS

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Abstract: The paper develops design methods for a class of uncertain, non-linear control systems. It extends the robustness analysis techniques of linear time varying systems and in particular the associated computational methodology to a class of nonlinear systems. It can be divided into following four parts: theoretical background, identification procedure, structure of feedback control system and cost functional for control optimisation for uncertain non-linear systems, control optimisation algorithm and the method for estimate the worst case output uncertainty norm of the system.

Keywords: Uncertain systems, non-linear systems, discrete-time systems, non-linear control design, non-linear optimisation.

1 INTRODUCTION

Analysis and control synthesis for linear uncertain systems or systems with limited information is wide area of scientific and engineering interests. The method presented in the paper extend the robustness analysis techniques of linear time varying systems and in particular the associated computational methodology to a class of nonlinear systems.

It is difficult to give an analytical methods for nonlinear, uncertain control synthesis. Very often one uses some simplifications which allow to make use of existing methods. In the last decade a lot of research achievements have been made on robust control design. The literature can be classified into two categories: Eigenstructure assignment and Riccati-based methods such as H_2, H_∞ and μ synthesis. Other works focus on simplification the nonlinear system, e.g. describing function analysis and linearization.

2 MODEL OF NONLINEAR SYSTEM

Non-linear system can be in general described by following model:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) \\ \mathbf{y}_k &= \mathbf{g}\left(\mathbf{x}_k\right) \end{aligned} \tag{1}$$

Non-linear coefficients state space model can be written as follows

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}(\mathbf{x}_k)\cdot\mathbf{x}_k + \mathbf{B}(\mathbf{u}_k)\cdot\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}(\mathbf{x}_k)\cdot\mathbf{x}_k + \mathbf{D}(\mathbf{u}_k)\cdot\mathbf{u}_k \end{aligned} \tag{2}$$

An important property of real discrete-time control systems is non-zero delay between input and output, thus $\mathbf{D}(\mathbf{u}_k) = \mathbf{0}$ and the model has following form:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}(\mathbf{x}_k)\cdot\mathbf{x}_k + \mathbf{B}(\mathbf{u}_k)\cdot\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}(\mathbf{x}_k)\cdot\mathbf{x}_k \end{aligned} \tag{3}$$

Above description is analogous to classical linear state space model. Matrices' coefficients $\{a_{ij}\},\{b_{ij}\},\{c_{ij}\}$ can be arbitrary functions of the state $a_{ij} = f_{ij}(\mathbf{x}), c_{ij} = h_{ij}(\mathbf{x})$ or the input $b_{ij} = g_{ij}(\mathbf{u})$.

For good estimation using identified data one has to known relations $a_{ij} = f_{ij}(\mathbf{x}_j), c_{ij} = h_{ij}(\mathbf{x}_j)$ $b_{ij} = g_{ij}(\mathbf{u}_j)$ in the range of variation state and input with sufficient resolution.

3 MODEL OF UNCERTAINTY

It is assumed following uncertain, additive non-linear model of the system

$$\begin{aligned} \mathbf{x}^\Delta_{k+1} &= \mathbf{A}_\Delta(\mathbf{x}^\Delta_k)\cdot\mathbf{x}^\Delta_k + \mathbf{B}_\Delta(\mathbf{u}_k)\cdot\mathbf{u}_k \\ \mathbf{y}^\Delta_k &= \mathbf{C}_\Delta(\mathbf{x}^\Delta_k)\cdot\mathbf{x}^\Delta_k \end{aligned} \tag{4}$$

Uncertain system produces uncertain state $\mathbf{x}^{\Delta}$ and uncertain output $\mathbf{y}^{\Delta}$. In general they are different from nominal state $\mathbf{x}^{p}$ and nominal output $\mathbf{y}^{p}$, thus $\mathbf{y}_k^{\Delta} \neq \mathbf{y}_k^{p}$ $\mathbf{x}_k^{\Delta} \neq \mathbf{x}_k^{p}$ and $\mathbf{A}(\mathbf{x}^{p}) \neq \mathbf{A}(\mathbf{x}^{\Delta}),\ \mathbf{A}_{\Delta}(\mathbf{x}^{p}) \neq \mathbf{A}_{\Delta}(\mathbf{x}^{\Delta})$. Non-linear matrices' functions can be expanded in multidimensional Taylor series, for matrix **A** it is

$$\mathbf{A}_{\Delta}(\mathbf{x}^{\Delta}) = \mathbf{A}_{\Delta}(\mathbf{x}^{p}) + \frac{\mathbf{A}_{\Delta}^{'}(\mathbf{x}^{p})}{1!}\cdot\left(\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right) + \frac{\mathbf{A}_{\Delta}^{''}(\mathbf{x}^{p})}{2!}\cdot\left(\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right)^{2} + \ldots \tag{5}$$

When the error state trajectory is small enough $\left\|\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right\| << 1$, the series is convergent and it is possible to rewrite it in the form

$$\mathbf{A}_{\Delta}(\mathbf{x}^{\Delta}) = \mathbf{A}_{\Delta}(\mathbf{x}^{p}) + \Delta_{\mathbf{A}r}\cdot\left(\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right) \tag{6}$$

where $\Delta_{\mathrm{A}r}$ satisfy the conditions

$$\Delta_{\mathbf{A}r} = \frac{\mathbf{A}_{\Delta}^{'}(\mathbf{x}^{p})}{1!} + \frac{\mathbf{A}_{\Delta}^{''}(\mathbf{x}^{p})}{2!}\cdot\left(\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right) + \ldots \tag{7}$$

$$\mathbf{A}_{\Delta}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \Delta_{\mathbf{A}}(\mathbf{x}) \tag{8}$$

Finally all system matrices have the form

$$\mathbf{A}_{\Delta}(\mathbf{x}^{\Delta}) = \mathbf{A}(\mathbf{x}^{p}) + \Delta_{\mathbf{A}}(\mathbf{x}^{p}) + \Delta_{\mathbf{A}r}(\mathbf{x}^{p})\cdot\left(\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right) \tag{9}$$

$$\mathbf{B}_{\Delta}(\mathbf{u}^{\Delta}) = \mathbf{B}(\mathbf{u}^{p}) + \Delta_{\mathbf{B}}(\mathbf{u}^{p}) + \Delta_{\mathbf{B}r}(\mathbf{u}^{p})\cdot\left(\mathbf{u}^{\Delta}-\mathbf{u}^{p}\right) \tag{10}$$

$$\mathbf{C}_{\Delta}(\mathbf{x}^{\Delta}) = \mathbf{C}(\mathbf{x}^{p}) + \Delta_{\mathbf{C}}(\mathbf{x}^{p}) + \Delta_{\mathbf{C}r}(\mathbf{x}^{p})\cdot\left(\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right) \tag{11}$$

The model of uncertainty for any perturbed system matrix **A, B, C** in uncertain state consists of three components:

- corresponding to nominal matrix in nominal state (or input), e.g. $\mathbf{A}(x^p)$,
- additive perturbation in nominal state (or input), e.g. $\Delta_{\mathbf{A}}(\mathbf{x}^{p})$, which doesn't depend on deviation from nominal state (or input),
- differential perturbation in nominal state (or input), e.g. $\Delta_{\mathbf{A}\mathrm{r}}(\mathbf{x}^{p})$, which represents only increase uncertainty in connection with the state (or input) deviation.

Of course it is not possible to find the additive $\Delta_{\mathbf{A}}$, $\Delta_{\mathbf{B}}$, $\Delta_{\mathbf{C}}$, and the differential $\Delta_{\mathbf{A}r}$, $\Delta_{\mathbf{B}r}$, $\Delta_{\mathbf{C}r}$, perturbation matrices, but it is possible to find their estimates δ_A, δ_B, δ_C, δ_{Ar}, δ_{Br}, δ_{Cr}, such that following conditions are held

for matrix **A**

$$\left\|\Delta_{\mathbf{A}}(\mathbf{x}_k^{p})\right\| \leq \delta_A < \infty \tag{12}$$

$$\left\|\Delta_{\mathbf{A}r}(\mathbf{x}_k^{p})\right\| \leq \delta_{Ar} < \infty \tag{13}$$

where $\Delta_{\mathrm{A}}(\mathbf{x}_k^{p}) \in \mathcal{L}(\mathbf{R}^{n},\mathbf{R}^{n})$, $\Delta_{\mathrm{A}r}(\mathbf{x}_k^{p}) \in \mathcal{L}(\mathbf{R}^{n},\mathbf{R}^{n})$.

and for matrices **B** and **C**:

$$\left\|\Delta_{\mathbf{B}}(\mathbf{u}_k^{p})\right\| \leq \delta_B < \infty \tag{14}$$

$$\left\|\Delta_{\mathbf{B}r}(\mathbf{u}_k^{p})\right\| \leq \delta_{Br} < \infty \tag{15}$$

$$\left\|\Delta_{\mathbf{C}}(\mathbf{x}_k^{p})\right\| \leq \delta_C < \infty \tag{16}$$

$$\left\|\Delta_{\mathbf{C}r}(\mathbf{x}_k^{p})\right\| \leq \delta_{Cr} < \infty \tag{17}$$

where $\Delta_{\mathbf{B}}(\mathbf{u}_k^{p}) \in \mathcal{L}(\mathbf{R}^{m},\mathbf{R}^{n})$, $\Delta_{\mathbf{B}r}(\mathbf{u}_k^{p}) \in \mathcal{L}(\mathbf{R}^{m},\mathbf{R}^{n})$, $\Delta_{\mathbf{C}}(\mathbf{x}_k^{p}) \in \mathcal{L}(\mathbf{R}^{n},\mathbf{R}^{p})$, $\Delta_{\mathbf{C}r}(\mathbf{x}_k^{p}) \in \mathcal{L}(\mathbf{R}^{n},\mathbf{R}^{p})$, $k=0,1,\ldots,N-1$.

4 IDENTIFICATION

Identification of matrices $\mathbf{A}(\mathbf{x}),\mathbf{B}(\mathbf{u}),\mathbf{C}(\mathbf{x})$ can be carried out using methods designed for linear systems under condition, that the system does not change the working point during identification. It mean variation of state and input must be small or slowly.

Choose of the working points for identification can depends on system (in some cases it is not possible to force specified conditions) or on the identifier. In the second case optimal set of the points for identification can be obtained using classical methods of experimental design e.g. factorial, simplex design etc.

4.1 Estimation of uncertainty in identified model

Although estimation of fundamental parameters of distribution is good known, estimation of uncertainty for dynamical systems parameters obtained from identification is more difficult. One possibility of estimation parameters is to carry through when identification is done repeatedly in every point of work.

The model of the system is given by equation (4). In assumption of rectangular or normal distribution of uncertainty, it is possible to estimate nominal model's parameters as the mean of realisations

$$\left\{a_{ij}^{n}\right\} = \frac{1}{N}\cdot\sum_{r=1}^{N} a_{ij}^{r} \tag{18}$$

where N denotes the number of realisations the identification in any given working point and for high accuracy should be large. Index n denotes nominal value of i,j- coefficient of matrix **A**. Index r denotes value of the coefficient in r realisation (in r process of identification).

The measure of uncertainty depends on distribution. For normal distribution the measure of uncertainty can be obtained using standard statistical procedure, which can be calculated from following relations

$$\left\{\delta_{ij}^{a}\right\} = \frac{\varepsilon_{\alpha}}{\sqrt{N-1}} \cdot \sqrt{\frac{1}{N} \cdot \sum_{r=1}^{N} \left(a_{ij}^{r} - a_{ij}^{n}\right)^{2}} \qquad (19)$$

where ε_{α} is parameter obtained from Student's distribution.

For rectangular distribution the measure of uncertainty is maximal deviation which can be obtained using following equation

$$\left\{\delta_{ij}^{a}\right\} = \max_{1 \le r \le N}\left(\left|a_{ij}^{r}\right| - \left|a_{ij}^{n}\right|\right) \qquad (20)$$

Using equation (18) also for coefficients of matrices **B** and **C,** it is possible to estimate the nominal matrix functions for all system matrices $\mathbf{A}(\mathbf{x}^{p})$, $\mathbf{B}(\mathbf{u}^{p})$, $\mathbf{C}(\mathbf{x}^{p})$.

Using equations (19) and (20) to matrices $\Delta^{\mathbf{a}}(\mathbf{x}) = \left\{\delta_{ij}^{a}(\mathbf{x}_{j})\right\}, \Delta^{\mathbf{b}}(\mathbf{x}) = \left\{\delta_{ij}^{b}(\mathbf{x}_{j})\right\}, \Delta^{\mathbf{c}}(\mathbf{x}) = \left\{\delta_{ij}^{c}(\mathbf{x}_{j})\right\}$ allow to estimate additive perturbations δ_A, δ_B, δ_C, given by equations (12), (14), (16). Relation between matrices $\Delta^{\mathbf{a}}(\mathbf{x}) = \left\{\delta_{ij}^{a}(\mathbf{x}_{j})\right\}$ and $\Delta_{\mathbf{A}}(\mathbf{x}) = \left\{\delta_{ij}^{A}(\mathbf{x}_{j})\right\}$ is following

$$\left|\delta_{ij}^{A}(\mathbf{x}_{j})\right| \le \delta_{ij}^{a}(\mathbf{x}_{j}) \text{ for all } i \text{ and } j, \qquad (21)$$

and

$$\begin{aligned} \max_{\delta_{ij}^{A}}\left(\left\|\Delta_{\mathbf{A}}(\mathbf{x})\right\|\right) &\approx \left\|\Delta^{\mathbf{a}}(\mathbf{x})\right\| = \delta^{\mathbf{A}a}(\mathbf{x}) \\ \delta_{A} &= \max_{\mathbf{x}} \delta^{\mathbf{A}a}(\mathbf{x}) \end{aligned} \qquad (22)$$

Similar relations are held for matrices $\Delta^{\mathbf{b}}(\mathbf{x})$, $\Delta_{\mathbf{B}}(\mathbf{x})$, $\Delta^{\mathbf{c}}(\mathbf{x})$, $\Delta_{\mathbf{C}}(\mathbf{x})$.

Estimates for differential perturbations δ_{Ar}, δ_{Br}, δ_{Cr}, can be obtained calculating differences between matrices identified for different working points. Following relations allow to obtain the estimates of differential perturbations

$$\delta_{Ar} = \max_{i,j,i\neq j} \frac{\left\|\mathbf{A}(\mathbf{x}_{i}^{p}) - \mathbf{A}(\mathbf{x}_{j}^{p})\right\| + \left|\delta^{\mathbf{A}}(\mathbf{x}_{i}) - \delta^{\mathbf{A}}(\mathbf{x}_{j})\right|}{\left\|\mathbf{x}_{i}^{p} - \mathbf{x}_{j}^{p}\right\|} \qquad (23)$$

$$\delta_{Br} = \max_{i,j,i\neq j} \frac{\left\|\mathbf{B}(\mathbf{u}_{i}^{p}) - \mathbf{B}(\mathbf{u}_{j}^{p})\right\| + \left|\delta^{\mathbf{B}}(\mathbf{u}_{i}) - \delta^{\mathbf{B}}(\mathbf{u}_{j})\right|}{\left\|\mathbf{u}_{i}^{p} - \mathbf{u}_{j}^{p}\right\|} \qquad (24)$$

$$\delta_{Cr} = \max_{i,j,i\neq j} \frac{\left\|\mathbf{C}(\mathbf{x}_{i}^{p}) - \mathbf{C}(\mathbf{x}_{j}^{p})\right\| + \left|\delta^{\mathbf{C}}(\mathbf{x}_{i}) - \delta^{\mathbf{C}}(\mathbf{x}_{j})\right|}{\left\|\mathbf{x}_{i}^{p} - \mathbf{x}_{j}^{p}\right\|} \qquad (25)$$

5 CONTROL SYSTEM

Control systems most often have two components: linear or non-linear controller and actuator, often non-linear. In contradistinction to the plant, the model of controller and actuator is known.

Literature about robust control system's synthesis is divided to a few parts: linear feedback controller design using optimisation or performance analysis methods and non-linear control design such as fuzzy-logic methods.

5.1 Mathematical description of control

The most general description which can cover many different designed controllers can be written in non-linear state feedback operator or matrix form $\mathbf{F}(\mathbf{x}_k)$.

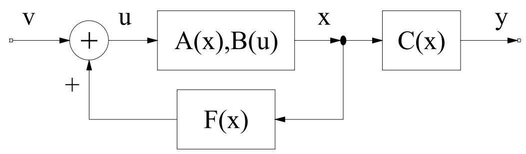

Fig. 1. Feedback control loop

Such operator can describe linear controller (in this case the coefficients are constant) or non-linear control (coefficients are functions of state vector). The control is designed when the functions $\left\{f_{ij}(\mathbf{x})\right\}$ are known and satisfy the design requirements. Feedback operator **F** can be also dynamical operator. Block diagram for control loop is shown on figure 5. Calculating norm of non-linear dynamical feedback operator can be difficult. An example of description for such system is presented in next section.

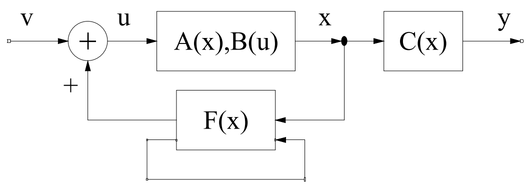

Fig. 2. Control loop with dynamical feedback

There are considered the case, where the form of the function $\mathbf{F}(\mathbf{x}) = \left\{f_{ij}(\mathbf{x})\right\}$ is assumed and one has to find only the values of coefficients.

5.2 Closed loop model

Input signal can be written as follows

$$\mathbf{u}_{k} = \mathbf{v}_{k} + \mathbf{F}(\mathbf{x}_{k}) \cdot \mathbf{x}_{k} \qquad (26)$$

where $\mathbf{F} \in \mathcal{L}\left(\mathbf{R}^{p}, \mathbf{R}^{m}\right)$ and k=0,1,...,N-1.

After substituting (26) into (3) the state space equations take the form

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}(\mathbf{x}_{k}) \cdot \mathbf{x}_{k} + \mathbf{B}(\mathbf{u}_{k}) \cdot \mathbf{F}(\mathbf{x}_{k}) \cdot \mathbf{x}_{k} + \mathbf{B}(\mathbf{u}_{k}) \cdot \mathbf{v}_{k} \\ \mathbf{y}_{k} &= \mathbf{C}(\mathbf{x}_{k}) \cdot \mathbf{x}_{k} \end{aligned} \qquad (27)$$

and simplifying

$$\begin{aligned} \mathbf{x}_{k+1} &= (\mathbf{A}(\mathbf{x}_{k}) + \mathbf{B}(\mathbf{u}_{k}) \cdot \mathbf{F}(\mathbf{x}_{k})) \cdot \mathbf{x}_{k} + \mathbf{B}(\mathbf{u}_{k}) \cdot \mathbf{v}_{k} \\ \mathbf{y}_{k} &= \mathbf{C}(\mathbf{x}_{k}) \cdot \mathbf{x}_{k} \qquad k = 0,1,\dots,N-1 \end{aligned} \qquad (28)$$

where $\mathbf{u}_k$ is given by equation (26).

5.3 Control law

It is difficult to proof analytical methods for non-linear, robust control design. Very often for control synthesis for such a systems one uses neural networks, genetic algorithms or finite element method. For using any optimisation method it is required to give the cost functional.

One of possibilities is to assume, that the cost functional is the norm of trajectory deviation for output of uncertain system in respect to given reference trajectory. In general it can be written following

$$J = \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{r}(\cdot) \right\| \tag{29}$$

Using following triangle inequality

$$\begin{aligned} &\left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{r}(\cdot) \right\| = \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) + \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\| \le \\ &\le \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\| + \left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\| \end{aligned} \tag{30}$$

it is possible to rewrite the functional in the form

$$J = \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\| + \left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\| \tag{31}$$

Nominal output deviation norm $\left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\|$ can be easily obtained, for example from simulation, output uncertainty norm $\left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\|$ can be only estimated. Because of conservatism in estimates it is possible to introduce weights in the cost functional. Finally, in particular the functional can be written for example in such a forms

$$J = \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\|_2 + \alpha \cdot \left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\|_2 \tag{32}$$

$$J = \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\|_\infty + \alpha \cdot \left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\|_\infty \tag{33}$$

$$J = \left\| \mathbf{y}^{\Delta}(N) - \mathbf{y}^{p}(N) \right\|_1 + \alpha \cdot \left\| \mathbf{y}^{p}(N) - \mathbf{y}^{r}(N) \right\|_1 \tag{34}$$

To find the coefficients, that minimise the cost functional can be employ one of existing methods for numerical, non-linear, multivariable optimisation.

6 ESTIMATE OF OUTPUT UNCERTAINTY NORM

There are three possibilities of calculating estimate of output uncertainty norm.

First possibility is to make some simulations of uncertain system for specified input and initial state in assumption of extreme positive and negative values of perturbation matrices $\Delta_{\mathbf{A}}(\mathbf{x})$. Maximal deviation norm of all simulations is an estimate of the norm $\left\| \mathbf{y}^{\Delta} - \mathbf{y}^{p} \right\|$. Number of simulations n_s grow exponentially with number of nonzero coefficients of additive perturbation, e.g. $n_s = 2^{nzcoeff}$. Extreme parameters' values do not guarantee calculating maximal deviation of output for every system. Although, the method is very useful for calculating such estimates.

Second possibility is to use optimisation method. The main disadvantage of the method is much more simulations required than for the first method. It is possible to use one of optimisation methods designed for multivariable optimisation e.g. genetic algorithm or simplex search. The number of variables is equal to sum of all nonzero coefficients of additive perturbations. Second method gives the most accurate estimates.

Third, operators' based method guarantee, that the estimated output difference norm is not lower than, the worst possible case. The main disadvantage is conservatism in estimates, in some case very large. The method can be useful also in other applications. Detailed description of the method is given in next section.

6.1 Description of non-linear feedback system using linear operators

Every linear time-varying system can be described by linear invariant, recurrent operators' equations. Very important property of non-linear system is repeatability. Every input and initial state sequence correspond to other invariant operators' equations, but for the same conditions equations are the same. Thus, non-linear system can be described by similar notation only in case of fixed input and initial state vector.

Let be $\mathbf{A}_k^{\mathbf{F}} = \mathbf{A}(\mathbf{x}_k) + \mathbf{B}(\mathbf{u}_k) \cdot \mathbf{F}(\mathbf{x}_k)$, $\mathbf{B}_k^{\mathbf{F}} = \mathbf{B}(\mathbf{u}_k)$, $\mathbf{C}_k^{\mathbf{F}} = \mathbf{C}(\mathbf{x}_k)$, then following operators can be defined for any fixed conditions $\mathbf{L}^{\mathbf{F}} \in \mathcal{L}((\mathbf{R}^n)^N, (\mathbf{R}^n)^N)$ and $\mathbf{N}^{\mathbf{F}} \in \mathcal{L}(\mathbf{R}^n, (\mathbf{R}^n)^N)$

$$\begin{aligned} &(\mathbf{L}^{\mathbf{F}}(\mathbf{B} \cdot \mathbf{v}))(k) = \\ &= \sum_{i=0}^{k-2} \left[\mathbf{A}_{k-1}^{\mathbf{F}} \cdot \mathbf{A}_{k-2}^{\mathbf{F}} \cdot \ldots \cdot \mathbf{A}_{i+1}^{\mathbf{F}} \cdot \mathbf{B}_i^{\mathbf{F}} \cdot \mathbf{v}_i \right] + \mathbf{B}_{k-1}^{\mathbf{F}} \cdot \mathbf{v}_{k-1} \end{aligned} \tag{35}$$

$$(\mathbf{N}^{\mathbf{F}} \mathbf{x}_0)(k) = \mathbf{A}_{k-1}^{\mathbf{F}} \cdot \mathbf{A}_{k-2}^{\mathbf{F}} \cdot \ldots \cdot \mathbf{A}_1^{\mathbf{F}} \cdot \mathbf{A}_0^{\mathbf{F}} \cdot \mathbf{x}_0 \tag{36}$$

where k=2,3,...,N.

6.1.1 Operators' equations for nominal system

Nominal non-linear system described by equations (28) and fixed input and initial state can be described by following equations

$$\begin{aligned} \mathbf{x}_k^p &= (\mathbf{N}^{\mathbf{F}} \mathbf{x}_0)(k) + (\mathbf{L}^{\mathbf{F}}(\mathbf{B} \cdot \mathbf{v}))(k) \\ \mathbf{y}_k^p &= \mathbf{C}_k^{\mathbf{F}} \cdot \mathbf{x}_k^p \end{aligned} \tag{37}$$

Above equations can be proofed using mathematical induction.

6.1.2 Operators' equations for perturbed system

Perturbed non-linear system described by equations (4) with feedback control (26) and fixed input and initial state is almost equal to following equations

$$\begin{aligned} \mathbf{x}_k^{\Delta} &= \mathbf{x}_k^{p} + \mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathbf{A}}(\mathbf{x}^{p})\cdot\mathbf{x}^{\Delta}\right)(k) \\ &+\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathbf{A}r}(\mathbf{x}^{p})\cdot\left(\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right)\right)(k) \\ &+\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathbf{B}}(\mathbf{u}^{p})\cdot(\mathbf{v}+\mathbf{F}(\mathbf{x}^{\Delta})\cdot\mathbf{x}^{\Delta}\right)(k) \\ &+\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathbf{B}r}(\mathbf{u}^{p})\cdot\left(\mathbf{F}(\mathbf{x}^{\Delta})\cdot\mathbf{x}^{\Delta}-\mathbf{F}(\mathbf{x}^{p})\cdot\mathbf{x}^{p}\right)\right)(k) \end{aligned} \tag{38}$$

$$\mathbf{y}_k^{\Delta}(k) = \mathbf{C}_k^{\mathbf{F}}\cdot\mathbf{x}_k^{\Delta} + \Delta_{\mathrm{C}}(\mathbf{x}_k^{p})\cdot\mathbf{x}_k^{\Delta} + \Delta'_{\mathrm{Cr}}(\mathbf{x}_k^{p})\cdot\left(\mathbf{x}_k^{\Delta}-\mathbf{x}_k^{p}\right) \tag{39}$$

Above equations can be proofed using mathematical induction.

6.2 Estimation

Using operator's notation it is possible to derive a formula for estimate norm of $\left\|\mathbf{y}^{\Delta}(\cdot)-\mathbf{y}^{p}(\cdot)\right\|$. Physical interpretation of the norm depends on the type of norm it can be 2-norm, ∞-norm or 1-norm.

Theorem 1. *For every additive* $\Delta_{\mathbf{A}}$, $\Delta_{\mathbf{B}}$, Δ_{C}, *and differential* $\Delta_{\mathbf{A}r}$, $\Delta_{\mathbf{B}r}$, $\Delta_{\mathbf{C}r}$, *perturbations, with conditions (12)-(17) and*

$$\left(\delta_{\mathrm{Axz}}+\delta_{\mathrm{Arz}}\cdot\left\|\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right\|\right)<\left\|\mathbf{L}^{\mathbf{F}}\right\|^{-1} \tag{40}$$

$$\left(\delta_{\mathrm{Axz}}+\delta_{\mathrm{Arz}}\cdot\left\|\mathbf{x}^{p}\right\|+\delta_{\mathrm{ab}}\right)<\left\|\mathbf{L}^{\mathbf{F}}\right\|^{-1} \tag{41}$$

norms of differences $\left\|\mathbf{x}^{\Delta}(\cdot)-\mathbf{x}^{p}(\cdot)\right\|_{(\mathbf{R}^{n})^{N}}$ *and* $\left\|\mathbf{y}^{\Delta}(\cdot)-\mathbf{y}^{p}(\cdot)\right\|_{(\mathbf{R}^{p})^{N}}$ *are described by*

$$\left\|\mathbf{x}^{\Delta}(\cdot)-\mathbf{x}^{p}(\cdot)\right\|\le\frac{\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left(\delta_{\mathrm{aoz}}+\delta_{\mathrm{Axz}}\cdot\left\|\mathbf{x}^{p}\right\|\right)}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left(\delta_{\mathrm{Axz}}+\delta_{\mathrm{Arz}}\cdot\left\|\mathbf{x}^{p}\right\|+\delta_{\mathrm{ab}}\right)} \tag{42}$$

$$\begin{aligned} &\left\|\mathbf{y}^{\Delta}(\cdot)-\mathbf{y}^{p}(\cdot)\right\|\le\delta_{\mathrm{C}}\cdot\left\|\mathbf{x}_{p}\right\|+ \\ &\frac{\left[\ \left\|\mathbf{C}\cdot\mathbf{L}^{\mathbf{F}}\right\|+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot(\delta_{\mathrm{C}}+\delta_{\mathrm{Cr}})\right]\cdot\left[\delta_{\mathrm{aoz}}+\delta_{\mathrm{Axz}}\cdot\left\|\mathbf{x}_{p}\right\|\right]}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left(\delta_{\mathrm{Axz}}+\delta_{\mathrm{Arz}}\cdot\left\|\mathbf{x}_{p}\right\|+\delta_{\mathrm{ab}}\right)} \end{aligned} \tag{43}$$

where

$$\delta_{\mathrm{Axz}}=\delta_{\mathrm{A}}+\delta_{\mathrm{B}}\cdot\left\|\mathbf{F}\right\| \tag{44}$$

$$\delta_{\mathrm{Arz}}=\delta_{\mathrm{Ar}}+\delta_{\mathrm{Br}}\cdot\left\|\mathbf{F}\right\|^{2} \tag{45}$$

$$\delta_{\mathrm{aoz}}=\delta_{\mathrm{B}}\cdot\left\|\mathbf{v}\right\| \tag{46}$$

$$\delta_{\mathrm{ab}}=\delta_{\mathrm{Br}}\cdot\left\|\mathbf{v}\right\|\cdot\left\|\mathbf{F}\right\| \tag{47}$$

Proof of theorem 1 has been presented in Orlowski (2000/2).

7 CONTROL OPTIMISATION

Coefficients that minimise cost functional (31)-(34), can be find using existing optimisation methods. Multivariable optimisation methods have been described by (Gill et. all 1991), (Fletcher 1980), (Nelder, Mead 1965).

Although a wide spectrum of methods exists for unconstrained optimization, methods can be broadly categorized in terms of the derivative information that is, or is not, used. Search methods that use only function evaluations e.g., the simplex search (Nelder, Mead 1965) are most suitable for problems that are very nonlinear or have a number of discontinuities. Gradient methods are generally more efficient when the function to be minimized is continuous in its first derivative. Higher order methods, such as Newton's method, are only really suitable when the second order information is readily and easily calculated since calculation of second order information, using numerical differentiation, is computationally expensive.

Nevertheless, the complex nature of the problem to minimize and difficulties with computing differential, the method, which was used in practice for this problem is evolutionary genetic algorithm.

The problem can be formulated as follows. For given system, fixed reference signal $\mathbf{y}^{r}$, set of possible inputs $\mathbf{v}\in\mathbf{V}$ and form of feedback function $\mathbf{F}\left(\mathbf{x}_k,a_1,\ldots,a_M\right)$, find values $a_1,\ldots,a_M$ which minimise cost functional J (32)-(34). Because of nonlinearities, as the method for searching optimal coefficients have been chosen differential evolutionary genetic algorithm *DEGA* implemented in Matlab.

For specific form of the cost functional J one have to find suitable mutation and crossover coefficientsand the method which is used to converse cost functional J into quality functional Q.

7 NUMERICAL EXAMPLES

The system under consideration is churchbell control system. Block diagram is drawn on fig. 3. Mathematical model of the system can be find in Orlowski (1999).

It is assumed, that the feedback control function have the form:

$$\Delta t_{zal}=a\cdot\left|\varepsilon\right|^{b}\cdot sign\left(\varepsilon\right) \tag{48}$$

where ε is deviation from set value of inverse of maximal velocity of the bell in current cycle, Δt_{zal} is the working time of the drive in next cycle, a and b are the parameters of non-linear controller.

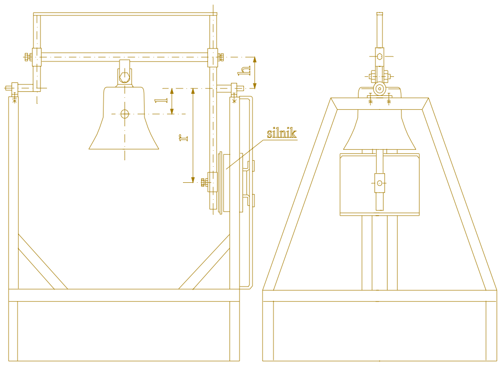

Fig. 3. Churchbell control system

Asssuming, that the cost functional J minimise sum of errors of acoustical energy in every stroke and the quality functional is given by the inverse $Q=1/J$, optimal parameters a, b estimated using *DEGA* function are following:

$$a=0.15,\ b=0.01 \text{ and } Q=25.37,\ J=0.0394$$

Output and input variables for churchbell control system with optimal coefficients is drawn on fig. 4.

8 CONCLUSION

For considered example of churchbell control system, the evolutionary genetic algorithm is convergent and can be use for such problems. Transformation of the cost functional into quality functional is quite important task for future results of optimisation. The function, which have been used in considered example of churchbell control system is inverse. Despite the large time for evaluation, this quality functional gives convergent results.

9 REFERENCES

Emirsajłow Z., Orłowski P. (1999). *„Analysis of finite horizon control problems for uncertain discrete-time system ", Proceedings of the 9th National Automation Conference*, Opole 21-24.IX.1999, vol. I, pp. 107-112 (in Polish)

Fletcher, R., "Practical Methods of Optimization," *Vol. 1, Unconstrained Optimization*, and *Vol. 2, Constrained Optimization,* John Wiley and Sons., 1980.

Gill, P.E., W. Murray, and M.H. Wright, *Numerical Linear Algebra and Optimization*, Vol. 1, Addison Wesley, 1991.

Nelder, J.A. and R. Mead, "A Simplex Method for Function Minimization", *Computer J.,* Vol .7, pp. 308-313, 1965.

Ogata, K. (1995). *Discrete-Time Control Systems*. Prentice Hall, Englewood Cliff, New Jersey.

Orłowski P., (1999). *Non-conventional bell's drive control with uncertain input,* III Scientific Symposium, Kazimierz Dolny 21-23.VI.1999, pp. 124-130 (in Polish)

Orłowski P., (2000/1). *Deviations estimates for uncertain time-varying discrete-time systems,* 3rd IFAC Symposium on Robust Control Design, Prague 21-23.VI.2000, Proceedings on CD-ROM art. no. 132.

Orłowski, P. (2000/2). *Selected Problems of Uncertainty Estimates for Non-linear Discrete-Time Systems*. OWD 2000, Istebna-Zaolzie.

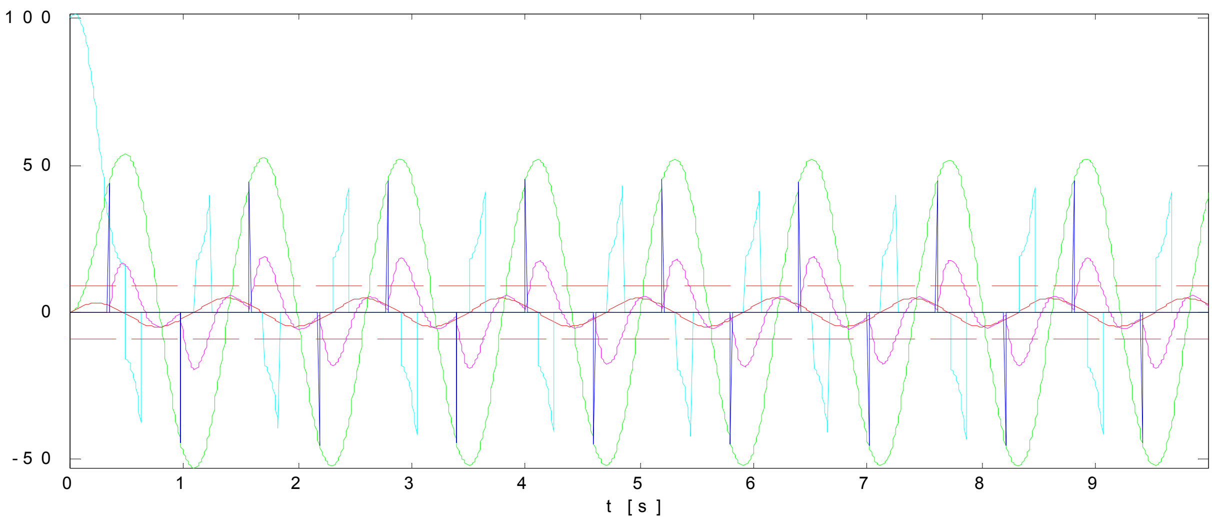

Fig. 4. Output and input variables for churchbell control system.