OUTPUT UNCERTAINTY ESTIMATES FOR A CLASS OF NON-LINEAR DISCRETE-TIME SYSTEMS

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Abstract. The paper develops a mathematical framework which helps to analyse a class of non-linear, uncertain, discrete-time systems defined on finite horizon control. Uncertainty in the system is modelled by unknown (norm bounded) additive perturbations of the system matrices. The main purpose of the paper is to derive estimates for output deviations system from the output of the nominal (unperturbed) one. These estimates use norms of certain dynamical operators defined on a finite time interval.

Keywords. Uncertain dynamic systems, uncertain linear systems, discrete-time systems, error estimation, non-linear systems, non-linear analysis, time-varying systems, state-space models.

1. INTRODUCTION

Control systems very often are uncertain, non-linear and/or time-varying. The most general models for such systems are given in state space. Synthesis of robust control require to know at least estimates of maximal differences between output signals among different conditions and uncertainties. Main purpose of this paper is to derive estimates and create tools for analysis for the most general case of state space model, with non-linear and time-varying coefficients. Troughout the paper it is assumed, that a positive integer N and a *nominal control* $\{\mathbf{u}_p(k)\in\mathbf{R}^m,\ k\in\{0,\dots,N-1\}\}$ which has feedback form. The corresponding *nominal state* and *nominal output* functions of Σ are denoted by $\mathbf{x}_p$ and $\mathbf{y}_p$. If the control is applied to the uncertain system then, in general $\mathbf{y}_\Delta\neq\mathbf{y}_p$. In particular the paper develops techniques for estimating the differences $\|\mathbf{y}_\Delta-\mathbf{y}_p\|$, $\|\mathbf{x}_\Delta-\mathbf{x}_p\|$ for a class of non-linear discrete-time systems on a finite time horizon.

2. METRIC SPACE

Space of vector's sequence are given by Hilbert space (l^2).

$$(\mathbf{R}^q)^N=\underbrace{\mathbf{R}^q\times\mathbf{R}^q\times\ldots\times\mathbf{R}^q}_{N\text{ times}}\tag{1}$$

Elements of the space are sequences of vectors

$$\mathbf{z}=[\mathbf{z}(0)\ldots\mathbf{z}(N-1)]^{\mathrm{T}}\tag{2}$$

where $\mathbf{z}(i)\in\mathbf{R}^q$, $i\in\{0,\dots,N-1\}$

Scalar product in the space is defined as follows

$$\langle\mathbf{z},\mathbf{v}\rangle_{(\mathbf{R}^q)^N}=\sum_{i=0}^{N-1}\langle\mathbf{z}(i),\mathbf{v}(i)\rangle_{\mathbf{R}^q}=\sum_{i=0}^{N-1}\mathbf{z}^{\mathrm{T}}(i)\cdot\mathbf{v}(i)\tag{3}$$

where $\mathbf{z},\mathbf{v}\in\mathbf{R}^q$, $i\in\{0,\dots,N-1\}$.

The induced norm has the form

$$\|\mathbf{z}\|^2_{(\mathbf{R}^q)^N}=\sum_{i=0}^{N-1}\langle\mathbf{z}(i),\mathbf{z}(i)\rangle_{\mathbf{R}^q}=\sum_{i=0}^{N-1}\mathbf{z}^{\mathrm{T}}(i)\cdot\mathbf{z}(i)\tag{4}$$

where $\mathbf{z}\in\mathbf{R}^q$, $i\in\{0,\dots,N-1\}$.

Space of matrix's sequences are given by Hilbert space.

$$(\mathbf{R}^p,\mathbf{R}^q)^N=\underbrace{(\mathbf{R}^p\times\mathbf{R}^q)\times\ldots\times(\mathbf{R}^p\times\mathbf{R}^q)}_{N\text{ times}}\tag{5}$$

Elements of the space are sequences of matrices

$$\mathbf{Z} = [\mathbf{Z}(0)...\mathbf{Z}(N-1)]^{\mathrm{T}} \tag{6}$$

where $\mathbf{Z}(i) \in \mathcal{L}(\mathbf{R}^{p}, \mathbf{R}^{q})$, $i \in \{0,...,N-1\}$

The most often used norms for vector spaces can be found e.g. in (Stewart, Sun 1990).

Generalised operator p-norm can be written as follows

$$\|\mathbf{A}\|_p = \sup_{\mathbf{h}} \frac{\|\mathbf{A}\cdot\mathbf{h}\|_p}{\|\mathbf{h}\|_p} \tag{7}$$

where **h** is sequence from input opearator space.

3. NOMINAL MODEL

The *nominal,* unperturbed, non-linear, control system is either

$$\mathbf{x}_p(k+1) = \mathbf{f}_0(\mathbf{x}_p(k), \mathbf{u}_p(k), k), \qquad \mathbf{x}_p(0)=\mathbf{x}_0, \tag{8}$$

or

$$\begin{aligned}\mathbf{x}_p(k+1) &= \mathbf{A}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_p(k)\\ &+\mathbf{B}(\mathbf{u}_p(k),k)\cdot\mathbf{u}_p(k)\end{aligned} \qquad \mathbf{x}_p(0)=\mathbf{x}_0, \tag{9}$$

$$\mathbf{y}_p(k) = \mathbf{C}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_p(k) \qquad k\in\mathbf{N}, \tag{10}$$

where $\{\mathbf{x}_p(k)\in\mathbf{R}^n,\ k\in\{0,...,N-1\}\}$ is nominal state, $\{\mathbf{u}_p(k)\in\mathbf{R}^m,\ k\in\{0,...,N-1\}\}$ is nominal control, $\{\mathbf{y}_p(k)\in\mathbf{R}^p,\ k\in\{0,...,N-1\}\}$ is nominal output, and { $\mathbf{A}(\mathbf{x}_p(k),k)\in\mathbf{R}^{n\times n}$, $\mathbf{B}(\mathbf{u}_p(k),k)\in\mathbf{R}^{n\times m}$, $\mathbf{C}(\mathbf{x}_p(k),k)\in\mathbf{R}^{p\times n}$ where $k\in\{0,...,N-1\}$ } are known matrices' functions. Function $\mathbf{f_0}$ is non-linear.

In order to cover the most general situation one can assume that control has the following feedback form

$$\mathbf{u}_p(k) = \mathbf{v}_p(k) + \mathbf{F}(k)\cdot\mathbf{x}_p(k) \tag{11}$$

where { $\mathbf{v}_p(k)\in\mathbf{R}^m$, $\mathbf{F}(k)\in\mathcal{L}(\mathbf{R}^p,\mathbf{R}^m)$ and k=0,1,...,N-1}.

Substituting (11) into state equations (9-10) gives

$$\begin{aligned}&\mathbf{x}_p(k+1) = (\mathbf{A}(\mathbf{x}_p(k),k) + \mathbf{B}(\mathbf{u}_p(k),k)\cdot\mathbf{F}(k))\\ &\cdot\mathbf{x}_p(k) + \mathbf{B}(\mathbf{u}_p(k),k)\cdot\mathbf{v}_p(k)\end{aligned} \tag{12}$$

$$\mathbf{y}_p(k) = \mathbf{C}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_p(k), \qquad \mathbf{x}_p(0)=\mathbf{x}_0, \quad k=0,1,...,N-1, \tag{13}$$

For the sake of simplicity it can be introduced three operators $\mathbf{L}^{\mathbf{F}} \in \mathcal{L}((\mathbf{R}^n)^N, (\mathbf{R}^n)^N)$, $\mathbf{K}^{\mathbf{F}} \in \mathcal{L}((\mathbf{R}^n)^N, \mathbf{R}^n)$ and $\mathbf{N}^{\mathbf{F}} \in \mathcal{L}(\mathbf{R}^n, (\mathbf{R}^n)^N)$, defined as follows

$$\begin{aligned}&(\mathbf{L}^{\mathbf{F}}(\mathbf{B}\cdot\mathbf{v}_p))(k) = \sum_{i=0}^{k-2}\left[\prod_{j=i+1}^{k-1}\left(\begin{aligned}&\mathbf{A}(\mathbf{x}_p(j),j)+\\ &\mathbf{B}(\mathbf{v}_p(j)+\mathbf{x}_p(j)\cdot\mathbf{F}(j),j)\cdot\mathbf{F}(j)\end{aligned}\right)\right]\\ &\cdot\mathbf{B}(\mathbf{v}_p(i)+\mathbf{x}_p(i)\cdot\mathbf{F}(i),i)\cdot\mathbf{v}_p(i)\\ &+\mathbf{B}(\mathbf{v}_p(k-1)+\mathbf{x}_p(k-1)\cdot\mathbf{F}(k-1),k-1)\cdot\mathbf{v}_p(k-1)\end{aligned} \tag{14}$$

$$\begin{aligned}&\mathbf{K}^{\mathbf{F}}(\mathbf{B}\cdot\mathbf{v}_p) = \sum_{i=0}^{N-2}\left[\prod_{j=i+1}^{N-1}\left(\begin{aligned}&\mathbf{A}(\mathbf{x}_p(j),j)+\\ &\mathbf{B}(\mathbf{v}_p(j)+\mathbf{x}_p(j)\cdot\mathbf{F}(j),j)\cdot\mathbf{F}(j)\end{aligned}\right)\right]\\ &\cdot\mathbf{B}(\mathbf{v}_p(i)+\mathbf{x}_p(i)\cdot\mathbf{F}(i),i)\cdot\mathbf{v}_p(i)+\mathbf{B}(\mathbf{v}_p(k-1)\\ &+\mathbf{x}_p(N-1)\cdot\mathbf{F}(N-1),N-1)\cdot\mathbf{v}_p(N-1)\end{aligned} \tag{15}$$

$$(\mathbf{N}^{\mathbf{F}}\mathbf{x}_0)(k) = \prod_{j=0}^{k-1}\left(\begin{aligned}&\mathbf{A}(\mathbf{x}_p(j),j)+\\ &\mathbf{B}(\mathbf{v}_p(j)+\mathbf{x}_p(j)\cdot\mathbf{F}(j),j)\cdot\mathbf{F}(j)\end{aligned}\right)\cdot\mathbf{x}_0 \tag{16}$$

where k=2,3,...,N.

Theorem 1. For every system Σ_p described by equations (9-10) the state and output trajectory can be written as follows

$$\mathbf{x}_p(k) = (\mathbf{N}^{\mathbf{F}}\mathbf{x}_0)(k) + (\mathbf{L}^{\mathbf{F}}(\mathbf{B}\cdot\mathbf{v}_p))(k) \tag{17}$$

$$\mathbf{y}_p(k) = \mathbf{C}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_p(k), \tag{18}$$

Proof:

Let be $\mathbf{A}_k^{\mathbf{F}} = \mathbf{A}(\mathbf{x}_p(k),k) + \mathbf{B}(\mathbf{u}_p(k),k)\cdot\mathbf{F}(k)$, $\mathbf{B}_k^{\mathbf{F}} = \mathbf{B}(\mathbf{u}_p(k),k)$, $\mathbf{C}_k^{\mathbf{F}} = \mathbf{C}(\mathbf{x}_p(k),k)$ and $\mathbf{x}_k^p = \mathbf{x}_p(k)$, $\mathbf{y}_k^p = \mathbf{y}_p(k)$, $\mathbf{v}_k = \mathbf{v}_p(k)$.

Then for k=2 state equations (17-18) and (9-10) are equal to

$$\mathbf{x}_2^p = \mathbf{A}_1^{\mathbf{F}}\cdot\mathbf{A}_0^{\mathbf{F}}\cdot\mathbf{x}_0 + \mathbf{A}_1^{\mathbf{F}}\cdot\mathbf{B}_0^{\mathbf{F}}\cdot\mathbf{v}_0 + \mathbf{B}_1^{\mathbf{F}}\cdot\mathbf{v}_1$$

Substituting (17-18) in (9-10) for k+1 it is

$$\begin{aligned}&\mathbf{x}_{k+1}^p = \mathbf{A}_k^{\mathbf{F}}\cdot\left(\mathbf{A}_{k-1}^{\mathbf{F}}\cdot\ldots\cdot\mathbf{A}_0^{\mathbf{F}}\cdot\mathbf{x}_0\right) + \mathbf{A}_k^{\mathbf{F}}\cdot\left(\mathbf{L}^{\mathbf{F}}\mathbf{B}^{\mathbf{F}}\mathbf{v}\right)(k)\\ &+\mathbf{B}_k^{\mathbf{F}}\cdot\mathbf{v}_k = \left(\mathbf{N}^{\mathbf{F}}\mathbf{x}_0\right)(k+1) + \left(\mathbf{L}^{\mathbf{F}}\mathbf{B}^{\mathbf{F}}\mathbf{v}\right)(k+1)\end{aligned}$$

What finish the proof.

4. PERTURBED MODEL

Real control system is different from (8) or (9-10) and may be described by *perturbed* models Σ_Δ. For the first case (8) the model is given following

$$\mathbf{x}_\Delta(k+1)=\mathbf{f}_{\Delta 0}(\mathbf{x}_\Delta(k),\mathbf{u}_\Delta(k),k),\ \mathbf{x}_\Delta(0)=\mathbf{x}_0, \quad (19)$$

$$\begin{aligned}&\mathbf{f}_{\Delta 0}(\mathbf{x}_\Delta(k),\mathbf{u}_\Delta(k),k)=\\&\mathbf{f}_0(\mathbf{x}_p(k),\mathbf{u}_p(k),k)+\Delta_{\mathrm{f0}}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)+\\&+\Delta'_{\mathrm{frx}}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)\cdot(\mathbf{x}_\Delta(k)-\mathbf{x}_p(k))\\&+\Delta'_{\mathrm{fru}}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)\cdot(\mathbf{u}_\Delta(k)-\mathbf{u}_p(k))\end{aligned} \quad (20)$$

where:

$\mathbf{f_0}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)\in\mathcal{L}(\mathbf{R}^n)$, $\Delta_{\mathbf{f0}}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)\in\mathcal{L}(\mathbf{R}^n)$, $\Delta'_{\mathbf{frx}}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^n)$, $\Delta'_{\mathbf{fru}}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^m)$, $k=0,1,\ldots,N-1$, and the three conditions have to be satisfied

$$\|\Delta_{\mathbf{f0}}(\mathbf{x}_p(\cdot),\mathbf{u}_p(\cdot),\cdot)\|\le\delta_{f0}<\infty, \quad (21)$$

$$\|\Delta'_{\mathbf{frx}}(\mathbf{x}_p(\cdot),\mathbf{u}_p(\cdot),\cdot)\|\le\delta_{frx}<\infty, \quad (22)$$

$$\|\Delta'_{\mathbf{fru}}(\mathbf{x}_p(\cdot),\mathbf{u}_p(\cdot),\cdot)\|\le\delta_{fru}<\infty, \quad (23)$$

For the second nominal model, described by (9-10), one can write following uncertain description

$$\mathbf{x}_\Delta(k+1)=\mathbf{A}_\Delta(\mathbf{x}_\Delta(k),k)\cdot\mathbf{x}_\Delta(k)+\mathbf{B}_\Delta(\mathbf{u}_\Delta(k),k)\cdot\mathbf{u}_\Delta(k)\quad \mathbf{x}_\Delta(0)=\mathbf{x}_0, \quad (24)$$

$$\mathbf{y}_\Delta(k)=\mathbf{C}_\Delta(\mathbf{x}_\Delta(k),k)\cdot\mathbf{x}_\Delta(k)\qquad k\in\mathbf{N}, \quad (25)$$

For matrix **A**

$$\begin{aligned}&\mathbf{A}_\Delta(\mathbf{x}_\Delta(k),k)\cdot\mathbf{x}_\Delta(k)=\mathbf{A}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_\Delta(k)\\&+\Delta_{\mathrm{A}}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_\Delta(k)\\&+\Delta'_{\mathrm{Ar}}(\mathbf{x}_p(k),k)\cdot(\mathbf{x}_\Delta(k)-\mathbf{x}_p(k))\end{aligned} \quad (26)$$

where $\Delta_{\mathbf{A}}(\mathbf{x}_p(k),k)\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^n)$, $\Delta'_{\mathbf{Ar}}(\mathbf{x}_p(k),k)\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^n)$, $k=0,1,\ldots,N-1$, and

$$\|\Delta_{\mathbf{A}}(\mathbf{x}_p(k),k)\|\le\delta_A<\infty, \quad (27)$$

$$\|\Delta'_{\mathbf{Ar}}(\mathbf{x}_p(k),k)\|\le\delta_{Ar}<\infty, \quad (28)$$

For matrix **B**

$$\begin{aligned}&\mathbf{B}_\Delta(\mathbf{u}_\Delta(k),k)\cdot\mathbf{u}_\Delta(k)=\mathbf{B}(\mathbf{u}_p(k),k)\cdot\mathbf{u}_\Delta(k)\\&+\Delta_{\mathrm{B}}(\mathbf{u}_p(k),k)\cdot\mathbf{u}_\Delta(k)\\&+\Delta'_{\mathrm{Br}}(\mathbf{u}_p(k),k)\cdot(\mathbf{u}_\Delta(k)-\mathbf{u}_p(k))\end{aligned} \quad (29)$$

where $\Delta_{\mathbf{B}}(\mathbf{u}_p(k),k)\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^m)$, $\Delta'_{\mathbf{Br}}(\mathbf{u}_p(k),k)\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^m)$, $k=0,1,\ldots,N-1$, and

$$\|\Delta_{\mathbf{B}}(\mathbf{u}_p(k),k)\|\le\delta_B<\infty, \quad (30)$$

$$\|\Delta'_{\mathbf{Br}}(\mathbf{u}_p(k),k)\|\le\delta_{Br}<\infty, \quad (31)$$

For matrix **C**

$$\begin{aligned}&\mathbf{C}_\Delta(\mathbf{x}_\Delta(k),k)\cdot\mathbf{x}_\Delta(k)=\mathbf{C}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_\Delta(k)+\\&+\Delta_{\mathrm{A}}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_\Delta(k)\\&+\Delta'_{\mathrm{Ar}}(\mathbf{x}_p(k),k)\cdot(\mathbf{x}_\Delta(k)-\mathbf{x}_p(k))\end{aligned} \quad (32)$$

where $\Delta_{\mathbf{C}}(\mathbf{x}_p(k),k)\in\mathcal{L}(\mathbf{R}^p,\mathbf{R}^n)$, $\Delta'_{\mathbf{Cr}}(\mathbf{x}_p(k),k)\in\mathcal{L}(\mathbf{R}^p,\mathbf{R}^n)$, $k=0,1,\ldots,N-1$, and

$$\|\Delta_{\mathbf{C}}(\mathbf{x}_p(k),k)\|\le\delta_C<\infty, \quad (33)$$

$$\|\Delta'_{\mathbf{Cr}}(\mathbf{x}_p(k),k)\|\le\delta_{Cr}<\infty, \quad (34)$$

To obtain the norm of maximal output deviation, one needn't to know the uncertainty matrices $\Delta_{\mathbf{A}}$, $\Delta_{\mathbf{B}}$, $\Delta_{\mathbf{C}}$, $\Delta'_{\mathbf{Ar}}$, $\Delta'_{\mathbf{Br}}$, $\Delta'_{\mathbf{Cr}}$, One has to known only the estimates δ_A, δ_B, δ_C, δ_{Ar}, δ_{Br}, δ_{Cr}.

Theorem 2. For every perturbed system Σ_Δ described by equations (24-25) the state and output trajectory can be written as follows

$$\begin{aligned}&\mathbf{x}_\Delta(k)=\mathbf{x}_p(k)+\mathbf{L^F}(\Delta_{\mathrm{A}}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_\Delta(k))(k)\\&+\mathbf{L^F}(\Delta'_{\mathrm{Ar}}(\mathbf{x}_p(k),k)\cdot(\mathbf{x}_\Delta(k)-\mathbf{x}_p(k)))(k)\\&+\mathbf{L^F}(\Delta_{\mathrm{B}}(\mathbf{v}_p(k)+\mathbf{F}(k)\cdot\mathbf{x}_p(k))\cdot(\mathbf{v}_\Delta(k)+\mathbf{F}(k)\cdot\mathbf{x}_\Delta(k))(k)\\&+\mathbf{L^F}(\Delta'_{\mathrm{Br}}(\mathbf{v}_p(k)+\mathbf{F}(k)\cdot\mathbf{x}_p(k))\\&\cdot(\mathbf{v}_\Delta(k)-\mathbf{v}_p(k)+\mathbf{F}(k)\cdot\mathbf{x}_\Delta(k)-\mathbf{F}(k)\cdot\mathbf{x}_p(k))(k)\end{aligned} \quad (35)$$

$$\begin{aligned}\mathbf{y}_\Delta(k)=\ &\mathbf{C}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_\Delta(k)+\Delta_{\mathrm{C}}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_\Delta(k)\\&+\Delta'_{\mathrm{Cr}}(\mathbf{x}_p(k),k)\cdot(\mathbf{x}_\Delta(k)-\mathbf{x}_p(k))\end{aligned} \quad (36)$$

Proof:

Above equations can be proofed using mathematical induction and substitution identical as in the proof of theorem 1.

For $k=2$ state equations (35-36) and (24-25) with feedback control (11) are equal to

$$\begin{aligned}\mathbf{x}_2^\Delta=\ &\mathbf{A}_1^{\mathbf{F}}\cdot\begin{bmatrix}\left(\mathbf{A}_0^{\mathbf{F}}+\Delta_{\mathbf{A}0}+\Delta_{\mathbf{B}0}\cdot\mathbf{F}_0\right)\cdot\mathbf{x}_0\\+\left(\mathbf{B}_0^{\mathbf{F}}+\Delta_{\mathbf{B}0}\right)\cdot\mathbf{v}_0\end{bmatrix}\\&+\left(\mathbf{B}_1^{\mathbf{F}}+\Delta_{\mathbf{B}1}+\Delta_{\mathbf{B}r1}\cdot\left(\mathbf{F}_1^\Delta\cdot\mathbf{x}_1^\Delta-\mathbf{F}_1^p\cdot\mathbf{x}_1^p\right)\right)\cdot\mathbf{v}_1\\&+\begin{pmatrix}\Delta_{\mathbf{A}1}+\Delta_{\mathbf{A}r1}\cdot\left(\mathbf{x}_1^\Delta-\mathbf{x}_1^p\right)+\Delta_{\mathbf{B}1}\cdot\mathbf{F}_1\\+\Delta_{\mathbf{B}r1}\cdot\left(\mathbf{F}_1^\Delta\cdot\mathbf{x}_1^\Delta-\mathbf{F}_1^p\cdot\mathbf{x}_1^p\right)\cdot\mathbf{F}_1^\Delta\end{pmatrix}\cdot\mathbf{x}_1^\Delta\end{aligned}$$

Substituting (35-36) in (24-25) for $k+1$ it is

$$\mathbf{x}_{k+1}^{\Delta}=\mathbf{B}_k^{\mathbf{F}\Delta}\cdot\mathbf{v}_k+\left(\mathbf{A}_k^{\mathbf{F}}+\Delta_{\mathbf{A}k}+\Delta_{\mathbf{B}k}\cdot\mathbf{F}_k\right)\cdot$$
$$\begin{bmatrix}\left(\mathbf{N}^{\mathbf{F}}\mathbf{x}_0\right)(k)+\left(\mathbf{L}^{\mathbf{F}}\mathbf{B}^{\mathbf{F}}\mathbf{v}\right)(k)+\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathbf{A}}\mathbf{x}^{\Delta}\right)(k)\\ +\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathbf{A}r}\cdot\left(\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right)\right)(k)+\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathrm{B}}(\mathbf{v}+\mathbf{F}\mathbf{x}^{\Delta}\right)(k)\\ +\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathrm{Br}}\left(\mathbf{F}\ \mathbf{x}^{\Delta}-\mathbf{F}\ \mathbf{x}^{p}\right)\right)(k)\end{bmatrix}$$
$$=\left(\mathbf{N}^{\mathbf{F}}\mathbf{x}_0\right)(k+1)+\left(\mathbf{L}^{\mathbf{F}}\mathbf{B}^{\mathbf{F}}\mathbf{v}\right)(k+1)+$$
$$+\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathbf{A}}\mathbf{x}^{\Delta}\right)(k+1)+\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathbf{A}r}\cdot\left(\mathbf{x}^{\Delta}-\mathbf{x}^{p}\right)\right)(k+1)$$
$$+\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathrm{B}}(\mathbf{v}+\mathbf{F}\mathbf{x}^{\Delta}\right)(k+1)$$
$$+\mathbf{L}^{\mathbf{F}}\left(\Delta_{\mathrm{Br}}\left(\mathbf{F}\ \mathbf{x}^{\Delta}-\mathbf{F}\ \mathbf{x}^{p}\right)\right)(k+1)$$

Detailed conversions are simple but laborious. Above is presented only sketch of proof.

5. TRAJECTORY DEVIATION NORM

Theorem 3. For every $\Delta_{\mathbf{A}}\in L(\mathbf{R}^n,\mathbf{R}^n)^N$, $\Delta_{\mathbf{B}}\in L(\mathbf{R}^n,\mathbf{R}^m)^N$, $\Delta_{\mathbf{C}}\in L(\mathbf{R}^p,\mathbf{R}^n)^N$, $\Delta'_{\mathbf{A}r}\in L(\mathbf{R}^n,\mathbf{R}^n)^N$, $\Delta'_{\mathbf{B}r}\in L(\mathbf{R}^n,\mathbf{R}^m)^N$, $\Delta'_{\mathbf{C}r}\in L(\mathbf{R}^p,\mathbf{R}^n)^N$ when equations (26-34) and

$$\left(\delta_{\mathrm{A}}+\delta_{\mathrm{B}}\cdot\|\mathbf{F}\|_{(\mathbf{R}^m,\mathbf{R}^n)^N}+\delta_{\mathrm{Ar}}+\delta_{\mathrm{Br}}\cdot\|\mathbf{F}\|_{(\mathbf{R}^m,\mathbf{R}^n)^N}\right)<\left\|\mathbf{L}^{\mathbf{F}}\right\|^{-1} \tag{37}$$

are satisfied, the distances $\|\mathbf{x}_\Delta(\cdot)-\mathbf{x}_p(\cdot)\|_{(\mathbf{R}^n)^N}$, $\|\mathbf{y}_\Delta(\cdot)-\mathbf{y}_p(\cdot)\|_{(\mathbf{R}^p)^N}$ for system (9-10) can be estimated as follows

$$\|\mathbf{x}_\Delta(\cdot)-\mathbf{x}_p(\cdot)\|_{(\mathbf{R}^n)^N}\le\frac{\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot(\delta_{\mathrm{aoz}}+\delta_{\mathrm{Axz}}\cdot\|\mathbf{x}_p\|)}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot(\delta_{\mathrm{Axz}}+\delta'_{\mathrm{Arz}})} \tag{38}$$

$$\|\mathbf{y}_\Delta(\cdot)-\mathbf{y}_p(\cdot)\|_{(\mathbf{R}^p)^N}\le\frac{\|\mathbf{x}_p\|+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\mathrm{aoz}}}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\mathrm{Axz}}} \tag{39}$$
$$+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta_{\mathrm{aoz}}+\delta_{\mathrm{Axz}}\cdot\|\mathbf{x}_p\|\right]\cdot$$
$$\frac{\left[\|\mathbf{C}\|+\delta'_{\mathrm{Cr}}+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta'_{\mathrm{Arz}}-(\|\mathbf{C}\|+\delta'_{\mathrm{Cr}})\cdot\delta_{\mathrm{Axz}}\right]\right]}{\left[1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\mathrm{Axz}}\right]\cdot\left[1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot(\delta_{\mathrm{Axz}}+\delta'_{\mathrm{Arz}})\right]}$$

where

$$\delta_{\mathrm{Axz}}=\delta_{\mathrm{Ax}}+\delta_{\mathrm{Bu}}\cdot\|\mathbf{F}\| \tag{40}$$
$$\delta'_{\mathrm{Arz}}=\delta'_{\mathrm{Ar}}+\delta'_{\mathrm{Br}}\cdot\|\mathbf{F}\| \tag{41}$$
$$\delta_{\mathrm{aoz}}=\delta_{\mathrm{Bu}}\cdot\|\mathbf{v}_\Delta\|+\delta'_{\mathrm{Br}}\cdot\|\mathbf{v}_\Delta-\mathbf{v}_p\| \tag{42}$$

Proof: It is a standard result of functional analysis, if (35) will be transformed with triangle inequality and (26-33) are satisfied, there is

$$\|\mathbf{x}_\Delta-\mathbf{x}_p\|_{(\mathbf{R}^n)^N}\le\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\mathrm{Ax}}\cdot\|\mathbf{x}_\Delta\|+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta'_{\mathrm{Ar}}\cdot\|\mathbf{x}_\Delta-\mathbf{x}_p\|$$
$$+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\mathrm{Bu}}\cdot\left[\|\mathbf{v}_\Delta\|+\|\mathbf{F}\|\cdot\|\mathbf{x}_\Delta\|\right]$$
$$+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta'_{\mathrm{Br}}\cdot\left[\|\mathbf{v}_\Delta-\mathbf{v}_p\|+\|\mathbf{F}\|\cdot\|\mathbf{x}_\Delta-\mathbf{x}_p\|\right]$$

then

$$\|\mathbf{x}_\Delta-\mathbf{x}_p\|_{(\mathbf{R}^n)^N}\le\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\begin{bmatrix}\delta_{\mathrm{Axz}}\cdot\|\mathbf{x}_\Delta\|+\delta_{\mathrm{aoz}}\\ +\delta'_{\mathrm{Arz}}\cdot\|\mathbf{x}_\Delta-\mathbf{x}_p\|\end{bmatrix}$$
$$\|\mathbf{x}_\Delta-\mathbf{x}_p\|_{(\mathbf{R}^n)^N}\cdot\left[1-\delta'_{\mathrm{Arz}}\cdot\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\|\mathbf{x}_\Delta-\mathbf{x}_p\|\right]\le$$
$$\le\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta_{\mathrm{Axz}}\cdot\|\mathbf{x}_\Delta\|+\delta_{\mathrm{aoz}}\right]$$
$$\|\mathbf{x}_\Delta-\mathbf{x}_p\|_{(\mathbf{R}^n)^N}\le\frac{\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta_{\mathrm{Axz}}\cdot\|\mathbf{x}_\Delta\|+\delta_{\mathrm{aoz}}\right]}{1-\delta'_{\mathrm{Arz}}\cdot\left\|\mathbf{L}^{\mathbf{F}}\right\|} \tag{43}$$
$$\|\mathbf{x}_\Delta\|_{(\mathbf{R}^n)^N}\le\|\mathbf{x}_p\|+$$
$$+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta_{\mathrm{Axz}}\cdot\|\mathbf{x}_\Delta\|+\delta'_{\mathrm{Arz}}\cdot\|\mathbf{x}_\Delta-\mathbf{x}_p\|+\delta_{\mathrm{aoz}}\right]$$
$$\|\mathbf{x}_\Delta\|_{(\mathbf{R}^n)^N}\le\frac{\|\mathbf{x}_p\|+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta'_{\mathrm{Arz}}\cdot\|\mathbf{x}_\Delta-\mathbf{x}_p\|+\delta_{\mathrm{aoz}}\right]}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\mathrm{Axz}}} \tag{44}$$

By substituting (44) into (43) one has the state trajectory estimate.

$$\|\mathbf{x}_\Delta-\mathbf{x}_p\|_{(\mathbf{R}^n)^N}\le$$
$$\frac{\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\mathrm{Axz}}\cdot\left[\|\mathbf{x}_p\|+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta'_{\mathrm{Arz}}\cdot\|\mathbf{x}_\Delta-\mathbf{x}_p\|+\delta_{\mathrm{aoz}}\right]\right]}{\left[1-\delta'_{\mathrm{Arz}}\cdot\left\|\mathbf{L}^{\mathbf{F}}\right\|\right]\cdot\left[1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\mathrm{Axz}}\right]}$$
$$+\frac{\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\mathrm{aoz}}}{1-\delta'_{\mathrm{Arz}}\cdot\left\|\mathbf{L}^{\mathbf{F}}\right\|} \tag{45}$$

After simplifying (45), and when equations (37) and (26-34) are satisfied the norm of uncertain state's deviation can be written as follows

$$\|\mathbf{x}_\Delta-\mathbf{x}_p\|_{(\mathbf{R}^n)^N}\le\frac{\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta_{\mathrm{Axz}}\cdot\|\mathbf{x}_p\|+\delta_{\mathrm{aoz}}\right]}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta_{\mathrm{Axz}}+\delta'_{\mathrm{Arz}}\right]}$$

It is equivalent to equation (38). Output difference $\mathbf{y}_\Delta$-$\mathbf{y}_p$ is given in the form

$$\mathbf{y}_\Delta(k)-\mathbf{y}_p(k)=\left[\mathbf{C}(\mathbf{x}_p(k),k)+\Delta'_{\mathrm{Cr}}(\mathbf{x}_p(k),k)\right]$$
$$\cdot(\mathbf{x}_\Delta(k)-\mathbf{x}_p(k))+\Delta_{\mathrm{C}}(\mathbf{x}_p(k),k)\cdot\mathbf{x}_\Delta(k) \tag{46}$$

After normalisation and using triangle inequality, the difference can be written as follows.

$$\left\|\mathbf{y}_{\Delta}(\cdot)-\mathbf{y}_{p}(\cdot)\right\|_{(\mathbf{R}^{p})^{N}} \leq \frac{\left\|\mathbf{x}_{p}\right\|+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\text{aoz}}}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\text{Axz}}}+$$

$$+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta_{\text{aoz}}+\delta_{\text{Axz}}\cdot\left\|\mathbf{x}_{p}\right\|\right]\cdot \tag{47}$$

$$\cdot\frac{\left[\left\|\mathbf{C}\right\|+\delta'_{\text{Cr}}+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left[\delta'_{\text{Arz}}-(\left\|\mathbf{C}\right\|+\delta'_{\text{Cr}})\cdot\delta_{\text{Axz}}\right]\right]}{\left[1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\text{Axz}}\right]\cdot\left[1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot(\delta_{\text{Axz}}+\delta'_{\text{Arz}})\right]}$$

What is equivalent to equation (39). □

Theorem 4. For every $\Delta_{\mathbf{f0}}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)\in\mathcal{L}(\mathbf{R}^n)$, $\Delta'_{\mathbf{frx}}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^n)$ and $\Delta'_{\mathbf{fru}}(\mathbf{x}_p(k),\mathbf{u}_p(k),k)\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^m)$, k=0,1,...,N-1, for system (8), when equations (20-23) are satisfied and following condition is hold

$$\delta_{frx}<1 \tag{48}$$

the difference norm $\left\|\mathbf{x}_{\Delta}(\cdot)-\mathbf{x}_{p}(\cdot)\right\|_{(\mathbf{R}^{n})^{N}}$ is described by

$$\left\|\mathbf{x}_{\Delta}(\cdot)-\mathbf{x}_{p}(\cdot)\right\|_{(\mathbf{R}^{n})^{N}} \leq$$

$$\leq\frac{\delta_{f0}+\delta_{fru}\cdot\left\|\mathbf{u}_{\Delta}(\cdot)-\mathbf{u}_{p}(\cdot)\right\|_{(\mathbf{R}^{m})^{N}}}{1-\delta_{frx}} \tag{49}$$

Proof

$$\begin{aligned}\mathbf{x}_{\Delta}(k)&=\mathbf{f}_{0}(\mathbf{x}_{p}(k),\mathbf{u}_{p}(k),k)+\Delta_{\text{f0}}(\mathbf{x}_{p}(k),\mathbf{u}_{p}(k),k)\\&+\Delta'_{\text{frx}}(\mathbf{x}_{p}(k),\mathbf{u}_{p}(k),k)\cdot(\mathbf{x}_{\Delta}(k)-\mathbf{x}_{p}(k))\\&+\Delta'_{\text{fru}}(\mathbf{x}_{p}(k),\mathbf{u}_{p}(k),k)\cdot(\mathbf{u}_{\Delta}(k)-\mathbf{u}_{p}(k))\end{aligned} \tag{50}$$

$$\begin{aligned}\mathbf{x}_{\Delta}(k)&=\mathbf{x}_{p}(k)+\Delta_{\text{f0}}(\mathbf{x}_{p}(k),\mathbf{u}_{p}(k),k)+\\&+\Delta'_{\text{frx}}(\mathbf{x}_{p}(k),\mathbf{u}_{p}(k),k)\cdot(\mathbf{x}_{\Delta}(k)-\mathbf{x}_{p}(k))\\&+\Delta'_{\text{fru}}(\mathbf{x}_{p}(k),\mathbf{u}_{p}(k),k)\cdot(\mathbf{u}_{\Delta}(k)-\mathbf{u}_{p}(k))\end{aligned} \tag{51}$$

$$\begin{aligned}\left\|\mathbf{x}_{\Delta}(\cdot)-\mathbf{x}_{p}(\cdot)\right\|_{(\mathbf{R}^{n})^{N}}&\leq\left\|\Delta_{\text{f0}}(\mathbf{x}_{p}(\cdot),\mathbf{u}_{p}(\cdot),\cdot)\right\|_{(\mathbf{R}^{n})^{N}}\\&+\left\|\Delta'_{\text{frx}}(\mathbf{x}_{p}(\cdot),\mathbf{u}_{p}(\cdot),\cdot)\cdot(\mathbf{x}_{\Delta}(\cdot)-\mathbf{x}_{p}(\cdot))\right\|_{(\mathbf{R}^{n})^{N}}\\&+\left\|\Delta'_{\text{fru}}(\mathbf{x}_{p}(\cdot),\mathbf{u}_{p}(\cdot),\cdot)\cdot(\mathbf{u}_{\Delta}(\cdot)-\mathbf{u}_{p}(\cdot))\right\|_{(\mathbf{R}^{n})^{N}}\end{aligned} \tag{52}$$

$$\left\|\mathbf{x}_{\Delta}(\cdot)-\mathbf{x}_{p}(\cdot)\right\|_{(\mathbf{R}^{n})^{N}}\leq\delta_{f0}+$$

$$\delta_{frx}\cdot\left\|(\mathbf{x}_{\Delta}(\cdot)-\mathbf{x}_{p}(\cdot))\right\|_{(\mathbf{R}^{n})^{N}}+\delta_{fru}\cdot\left\|(\mathbf{u}_{\Delta}(\cdot)-\mathbf{u}_{p}(\cdot))\right\|_{(\mathbf{R}^{n})^{N}} \tag{53}$$

$$\begin{aligned}&\left\|\mathbf{x}_{\Delta}(\cdot)-\mathbf{x}_{p}(\cdot)\right\|_{(\mathbf{R}^{n})^{N}}\cdot\left(1-\delta_{frx}\right)\leq\\&\delta_{f0}+\delta_{fru}\cdot\left\|(\mathbf{u}_{\Delta}(\cdot)-\mathbf{u}_{p}(\cdot))\right\|_{(\mathbf{R}^{n})^{N}}\end{aligned} \tag{54}$$

$$\left\|\mathbf{x}_{\Delta}(\cdot)-\mathbf{x}_{p}(\cdot)\right\|_{(\mathbf{R}^{n})^{N}}\leq\frac{\delta_{f0}+\delta_{fru}\cdot\left\|\mathbf{u}_{\Delta}(\cdot)-\mathbf{u}_{p}(\cdot)\right\|_{(\mathbf{R}^{m})^{N}}}{1-\delta_{frx}} \tag{55}$$

What finish the proof. □

Theorem 5. For every $\Delta_{\mathbf{A}}\in L(\mathbf{R}^n,\mathbf{R}^n)^N$, $\Delta_{\mathbf{B}}\in L(\mathbf{R}^n,\mathbf{R}^m)^N$, $\Delta_{\mathbf{C}}\in L(\mathbf{R}^p,\mathbf{R}^n)^N$, $\Delta'_{\mathbf{A}r}\in L(\mathbf{R}^n,\mathbf{R}^n)^N$, $\Delta'_{\mathbf{B}r}\in L(\mathbf{R}^n,\mathbf{R}^m)^N$, $\Delta'_{\mathbf{C}r}\in L(\mathbf{R}^p,\mathbf{R}^n)^N$, and system (8), when equations (20-23, 40-42) are satisfied the differences $\left\|\mathbf{x}_{\Delta}(N)-\mathbf{x}_{p}(N)\right\|_{\mathbf{R}^{n}}$, $\left\|\mathbf{y}_{\Delta}(N)-\mathbf{y}_{p}(N)\right\|_{\mathbf{R}^{p}}$ are described as follows

$$\left\|\mathbf{x}_{\Delta}(N)-\mathbf{x}_{p}(N)\right\|_{\mathbf{R}^{n}}\leq\frac{\left\|\mathbf{K}^{\mathbf{F}}\right\|\cdot(\delta_{\text{aoz}}+\delta_{\text{Axz}}\cdot\left\|\mathbf{x}_{p}\right\|)}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot(\delta_{\text{Axz}}+\delta'_{\text{Arz}})} \tag{56}$$

$$\left\|\mathbf{y}_{\Delta}(N)-\mathbf{y}_{p}(N)\right\|_{\mathbf{R}^{p}}\leq\frac{\left\|\mathbf{x}_{p}\right\|+\left\|\mathbf{K}^{\mathbf{F}}\right\|\cdot\delta_{\text{aoz}}}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\text{Axz}}}+$$

$$+\left\|\mathbf{K}^{\mathbf{F}}\right\|\cdot\left[\delta_{\text{aoz}}+\delta_{\text{Axz}}\cdot\left\|\mathbf{x}_{p}\right\|\right]\cdot$$

$$\frac{\left[\left\|\mathbf{C}\right\|+\delta'_{\text{Cr}}+\left\|\mathbf{K}^{\mathbf{F}}\right\|\cdot\left[\delta'_{\text{Arz}}-(\left\|\mathbf{C}\right\|+\delta'_{\text{Cr}})\cdot\delta_{\text{Axz}}\right]\right]}{\left[1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\text{Axz}}\right]\cdot\left[1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot(\delta_{\text{Axz}}+\delta'_{\text{Arz}})\right]} \tag{57}$$

Proof:

$$\begin{aligned}\left\|\mathbf{x}_{\Delta}-\mathbf{x}_{p}\right\|_{(\mathbf{R}^{n})^{N}}&\leq\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta'_{\text{Arz}}\cdot\left\|\mathbf{x}_{\Delta}-\mathbf{x}_{p}\right\|+\\&+\left\|\mathbf{K}^{\mathbf{F}}\right\|\cdot\left[\delta_{\text{Axz}}\cdot\left\|\mathbf{x}_{\Delta}\right\|+\delta_{\text{aoz}}\right]\end{aligned}$$

$$\begin{aligned}&\left\|\mathbf{x}_{\Delta}-\mathbf{x}_{p}\right\|_{(\mathbf{R}^{n})^{N}}\cdot\left[1-\delta'_{\text{Arz}}\cdot\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\left\|\mathbf{x}_{\Delta}-\mathbf{x}_{p}\right\|\right]\leq\\&\left\|\mathbf{K}^{\mathbf{F}}\right\|\cdot\left[\delta_{\text{Axz}}\cdot\left\|\mathbf{x}_{\Delta}\right\|+\delta_{\text{aoz}}\right]\end{aligned}$$

$$\left\|\mathbf{x}_{\Delta}-\mathbf{x}_{p}\right\|_{(\mathbf{R}^{n})^{N}}\leq\frac{\left\|\mathbf{K}^{\mathbf{F}}\right\|\cdot\left[\delta_{\text{Axz}}\cdot\left\|\mathbf{x}_{\Delta}\right\|+\delta_{\text{aoz}}\right]}{1-\delta'_{\text{Arz}}\cdot\left\|\mathbf{L}^{\mathbf{F}}\right\|} \tag{58}$$

$$\begin{aligned}\left\|\mathbf{x}_{\Delta}\right\|_{(\mathbf{R}^{n})^{N}}&\leq\left\|\mathbf{x}_{p}\right\|+\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\text{Axz}}\cdot\left\|\mathbf{x}_{\Delta}\right\|+\\&+\left\|\mathbf{K}^{\mathbf{F}}\right\|\cdot\left[\delta'_{\text{Arz}}\cdot\left\|\mathbf{x}_{\Delta}-\mathbf{x}_{p}\right\|+\delta_{\text{aoz}}\right]\end{aligned}$$

$$\left\|\mathbf{x}_{\Delta}\right\|_{(\mathbf{R}^{n})^{N}}\leq\frac{\left\|\mathbf{x}_{p}\right\|+\left\|\mathbf{K}^{\mathbf{F}}\right\|\cdot\left[\delta'_{\text{Arz}}\cdot\left\|\mathbf{x}_{\Delta}-\mathbf{x}_{p}\right\|+\delta_{\text{aoz}}\right]}{1-\left\|\mathbf{L}^{\mathbf{F}}\right\|\cdot\delta_{\text{Axz}}} \tag{59}$$

After substituting

$$\left\|\mathbf{x}_\Delta - \mathbf{x}_p\right\|_{\mathbf{R}^n} \le \frac{\left\|\mathbf{K}^\mathbf{F}\right\|\cdot\delta_{\mathrm{aoz}}}{1-\delta'_{\mathrm{Arz}}\cdot\left\|\mathbf{L}^\mathbf{F}\right\|} + \frac{\left\|\mathbf{K}^\mathbf{F}\right\|\cdot\delta_{\mathrm{Axz}}\cdot\left[\left\|\mathbf{x}_p\right\|+\left\|\mathbf{K}^\mathbf{F}\right\|\cdot\left[\delta'_{\mathrm{Arz}}\cdot\left\|\mathbf{x}_\Delta-\mathbf{x}_p\right\|+\delta_{\mathrm{aoz}}\right]\right]}{\left[1-\delta'_{\mathrm{Arz}}\cdot\left\|\mathbf{L}^\mathbf{F}\right\|\right]\cdot\left[1-\left\|\mathbf{L}^\mathbf{F}\right\|\cdot\delta_{\mathrm{Axz}}\right]} \tag{60}$$

and after simplification

$$\left\|\mathbf{x}_\Delta(N) - \mathbf{x}_p(N)\right\|_{\mathbf{R}^n} \le \frac{\left\|\mathbf{K}^\mathbf{F}\right\|\cdot(\delta_{\mathrm{aoz}}+\delta_{\mathrm{Axz}}\cdot\left\|\mathbf{x}_p\right\|)}{1-\left\|\mathbf{L}^\mathbf{F}\right\|\cdot(\delta_{\mathrm{Axz}}+\delta'_{\mathrm{Arz}})}\mathbb{Z} \tag{61}$$

The output difference $\mathbf{y}_\Delta$-$\mathbf{y}_\mathbf{p}$ is given by following equation

$$\mathbf{y}_\Delta(N) - \mathbf{y}_p(N) = \Delta_\mathrm{C}(\mathbf{x}_p(N),N)\cdot\mathbf{x}_\Delta(N) + \left[\mathbf{C}(\mathbf{x}_p(N),N)+\Delta'_{\mathrm{Cr}}(\mathbf{x}_p(N),N)\right]\cdot(\mathbf{x}_\Delta(N)-\mathbf{x}_p(N)) \tag{62}$$

After normalization and using triangle inequality it is

$$\left\|\mathbf{y}_\Delta(\cdot) - \mathbf{y}_p(\cdot)\right\|_{(\mathbf{R}^p)^N} \le \frac{\left\|\mathbf{x}_p\right\|+\left\|\mathbf{L}^\mathbf{F}\right\|\cdot\delta_{\mathrm{aoz}}}{1-\left\|\mathbf{L}^\mathbf{F}\right\|\cdot\delta_{\mathrm{Axz}}} + \\ +\left\|\mathbf{L}^\mathbf{F}\right\|\cdot\left[\delta_{\mathrm{aoz}}+\delta_{\mathrm{Axz}}\cdot\left\|\mathbf{x}_p\right\|\right]\cdot \\ \frac{\left[\left\|\mathbf{C}\right\|+\delta'_{\mathrm{Cr}}+\left\|\mathbf{L}^\mathbf{F}\right\|\cdot\left[\delta'_{\mathrm{Arz}}-(\left\|\mathbf{C}\right\|+\delta'_{\mathrm{Cr}})\cdot\delta_{\mathrm{Axz}}\right]\right]}{\left[1-\left\|\mathbf{L}^\mathbf{F}\right\|\cdot\delta_{\mathrm{Axz}}\right]\cdot\left[1-\left\|\mathbf{L}^\mathbf{F}\right\|\cdot(\delta_{\mathrm{Axz}}+\delta'_{\mathrm{Arz}})\right]}$$

What finish the proof of theorem 5. □

6. CONCLUSION

It follows from the above formulas that effectivness of the estimate (38) will highly depend on how good are the estimates of the operator norms $\|\mathbf{C}\cdot\mathbf{L}^\mathbf{F}\|$, $\|\mathbf{L}^\mathbf{F}\|$ etc.

Determining estimates for the general non-linear model (8) is very hard, because it is required to compute space estimates of (21-23). For the system with non-linear coefficients (9-10) it is easier, because the significant role play the method for determining norms of operators. The developed estimates can be used in various control tasks for both non-linear and time-varying uncertain discrete-time control systems.

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