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Periodic Linear Time-Varying System Norm Estimation Using Running Finite Time Horizon Transfer Operators

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Abstract: A novel method for norm estimation for dynamical linear time-varying systems is developed. The method involves operators description of the system model i.e. transfer operator. The transfer operator defined for finite time horizon can be described by finite dimensional matrix whereas for infinite time horizon the operator is infinite dimensional. The norm estimate for infinite time horizon is based on analysis of a running series of the finite time horizon norm properties.

Keywords: norm estimation, discrete-time systems, time-varying systems, non-stationary systems.

1. INTRODUCTION

In order to describe the dynamics of time-varying discrete-time systems, one can employ state space equations with time-dependent matrices given by eq. (1)-(2):

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{v}(k)\,, \tag{1}$$

$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{v}(k)\,,\ \mathbf{x}(k_0) = \mathbf{x}_0 \tag{2}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is nominal state, $\mathbf{v}(k) \in \mathbb{R}^m$ is the nominal control, $\mathbf{y}(k) \in \mathbb{R}^p$ is the nominal output and $\mathbf{A}(k) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(k) \in \mathbb{R}^{n\times m}$, $\mathbf{C}(k) \in \mathbb{R}^{p\times n}$, $\mathbf{D}(k) \in \mathbb{R}^{p\times m}$ are system matrices, $k = k_0, k_0+1, ..., k_0+N$ and N is length of the time horizon. For infinite time horizon $N = \infty$.

An LTV system can be equivalently described in terms of the matrix operators. There are two different approaches: one based on block diagonal operators Khalil (1996) and the other based on a lower triangular system matrix Orlowski (2004). Both approaches lead to an operator-based description of the system and a function which takes the role of a transfer function for time-varying systems. This function has many properties analogous to those of transfer functions of linear time-invariant (LTI) systems. In some cases, this allows one to apply to linear time-varying (LTV) systems techniques which have formerly been restricted to LTI systems.

Alternatively, the model may be described by means of operators. Equations (1)-(2) can be converted into following operators form:

$$\hat{\mathbf{y}} = \hat{\mathbf{C}}\hat{\mathbf{N}}\mathbf{x}_0 + \left(\hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}} + \hat{\mathbf{D}}\right)\hat{\mathbf{v}} = \hat{\mathbf{C}}\hat{\mathbf{N}}\mathbf{x}_0 + \hat{\mathbf{T}}\hat{\mathbf{v}} \tag{3}$$

In order that the system (3) be equivalent to the system (1)-(2), operators $\hat{\mathbf{T}} = \hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}} + \hat{\mathbf{D}}$ and $\hat{\mathbf{C}}\hat{\mathbf{N}}$ must be defined in one of the two equivalent notations: either an evolutionary one, where operators are written by means of sums and products Orlowski (2001):

$$\begin{aligned}\mathbf{y}(k) &= \left(\hat{\mathbf{C}}\hat{\mathbf{N}}\mathbf{x}_0\right)(k) + \left(\hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}}\hat{\mathbf{v}}\right)(k) + \mathbf{D}(k)\mathbf{v}(k) = \mathbf{C}(k)\phi_{k_0}^{k-1}\mathbf{x}_0 \\ &+\mathbf{C}(k)\left(\sum_{i=k_0}^{k-2}\phi_{i+1}^{k-1}\mathbf{B}(i)\mathbf{v}(i) + \mathbf{B}(k-1)\mathbf{v}(k-1)\right) + \mathbf{D}(k)\mathbf{v}(k)\end{aligned} \tag{4}$$

where $\phi_i^k = \mathbf{A}(k)\mathbf{A}(k-1)\ldots\mathbf{A}(i)$, or a matrix-based one, where each of the operators can be presented in terms of matrices. In order to analyze the stability of the system, one has to know operators $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ which can be expressed with the help of the following matrix operators:

$$\hat{\mathbf{L}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \phi_{k_0+1}^{k_0+1} & \mathbf{I} & \mathbf{0} & \vdots \\ \vdots & \ddots & \mathbf{I} & \mathbf{0} \\ \phi_{k_0+1}^{k_0+N-1} & \cdots & \phi_{k_0+N-1}^{k_0+N-1} & \mathbf{I} \end{bmatrix} \tag{5}$$

$$\hat{\mathbf{N}} = \begin{bmatrix} \phi_{k_0}^{k_0} \\ \vdots \\ \phi_{k_0}^{k_0+N-1} \end{bmatrix} \tag{6}$$

$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C}(k_0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}(k_0+N-1) \end{bmatrix} \tag{7}$$

Operator $\hat{\mathbf{N}}$ can be neglected when initial conditions are zero. Following sequences: state $\hat{\mathbf{x}}$, output $\hat{\mathbf{y}}$ and input $\hat{\mathbf{v}}$ are constructed from state $\mathbf{x}(k)$, output $\mathbf{y}(k)$ and input $\mathbf{v}(k)$ signals rewritten in following block column vector form:

$$\hat{\mathbf{x}} = \left[\mathbf{x}^T(k_0+1)\cdots\mathbf{x}^T(k_0+N)\right]^T \tag{8}$$

$$\hat{\mathbf{y}} = \left[\mathbf{y}^T(k_0+1)\cdots\mathbf{y}^T(k_0+N)\right]^T \tag{9}$$

$$\hat{\mathbf{v}} = \left[\mathbf{v}^T(k_0+1)\cdots\mathbf{v}^T(k_0+N)\right]^T \tag{10}$$

The input/output operator $\hat{\mathbf{T}}$ can be alternatively defined also using a set of impulse responses of a time-varying system taken at different times, e.g. for SISO system it may be written:

$$\hat{\mathbf{T}} = \begin{bmatrix} h_{k_0}^{k_0} & 0 & \cdots & 0 & 0 \\ h_{k_0}^{k_0+1} & h_{k_0+1}^{k_0+1} & \cdots & \vdots & \vdots \\ h_{k_0}^{k_0+2} & h_{k_0+1}^{k_0+2} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & h_{k_0+N-2}^{k_0+N-2} & 0 \\ h_{k_0}^{k_0+N-1} & \cdots & \cdots & h_{k_0+N-2}^{k_0+N-1} & h_{k_0+N-1}^{k_0+N-1} \end{bmatrix} \tag{11}$$

where $h_{k_0}^{k_1}$ is the response of the system to the Kronecker delta $\delta(k-k_0)$ at time k_1 (after k_1-k_0 samples). In the case of a nonzero input-output delay operator, $\hat{\mathbf{D}} = \mathbf{0}$ and all diagonal entries of $\hat{\mathbf{T}}$ are equal to zero.

For further considerations in the paper following definitions of norms for sequences and operators are used. The norm of a sequence in the Hilbert-space is understood as Euclidean norm:

$$\|\hat{\mathbf{v}}\| = \|\hat{\mathbf{v}}\|_2 = \sqrt{\langle\hat{\mathbf{v}},\hat{\mathbf{v}}\rangle} = \sum_k \mathbf{v}^{\mathrm{T}}(k)\,\mathbf{v}(k) = \hat{\mathbf{v}}^{\mathrm{T}}\hat{\mathbf{v}} \tag{12}$$

The ∞-norm of a sequence in the bounded sequences space is understood as:

$$\|\hat{\mathbf{v}}\|_\infty = \max_k\left(|v(k)|\right) \tag{13}$$

Norms of operators are defined in following way:

$$\|\hat{\mathbf{T}}\| = \|\hat{\mathbf{T}}\|_2 = \sup_{\hat{\mathbf{v}}\neq 0}\frac{\|\hat{\mathbf{T}}\hat{\mathbf{v}}\|_2}{\|\hat{\mathbf{v}}\|_2} \tag{14}$$

For systems defined on finite time horizon all operators are represented by finite dimensional matrices and signals by finite dimensional vectors. Moreover the input-output operator is a compact, Hilbert-Schmidt operator from l_2 into l_2 and actually maps bounded signals $v \in \mathcal{M} = l_2[k_0, k_0+N]$ into the signals $y \in \mathcal{P}$.

2. COMPUTATION THE NORM OF THE TIME-VARYING SYSTEM

Stability and performance criteria for analysis and robust control design of linear systems, are often expressed by norms of appropriately defined transfer functions or transfer operators, especially for time varying systems. Norms of the linear time-invariant systems defined on infinite time horizon can be easily computed using algorithms described in Bruisma et al. (1990), Bryson et al. (1975). The algorithms are also implemented in Matlab Control Toolbox Trefethen (2000). They needs only conversion of the system operator into state-space description. Although many methods for computing norms for linear time-invariant systems Boyd et al. (1990), Bruisma et al. (1990), Genin et al. (2002) which are essential in a computer aided control system design Zhou et al. (1995) there are very difficult to find methods applicable for linear time-varying systems.

Norm of transfer operator defined on infinite time horizon can be computed for periodic linear time-varying systems employing lifting technique. The paper (Bittanti et. al. 2000) is an overview and comparison of techniques which allows to rewrite time-varying systems using time-invariant representation with increased but finite dimensions. Norm of the transfer operator for such system can be computed in similar way as for linear time-invariant systems. More description for the lifting technique for periodic time-varying systems can be found in Bamieh et. al. (1991), Flamm (1991), Laub (1981), Meyer et al. (1975), Varga (1989).

Nevertheless norm of systems non periodic time varying systems cannot be easily computed. In such case the norm of transfer operator can be estimated using general operator theory Baladi et al. (1995), Descombes et al. (1999), Dewilde et al. (1993), Gohberg et al. (1984), Leblond et al. (1998) or the technique based on parameterised functional minimization. The main idea is based on the following general result given inOrlowski et al. (1999).

2.1. Parameterised functional based norm estimation

Theorem 1. *Let* $\mathcal{M}, \mathcal{P}$ *be real Hilbert spaces,* $\hat{\mathbf{T}} \in \mathcal{L}(\mathcal{M},\mathcal{P})$, $\hat{\mathbf{C}}\hat{\mathbf{N}} \in \mathcal{P}$, $\gamma \in (0,\infty)$ *and* $J(\hat{\mathbf{v}})$ *be a functional defined on* $\mathcal{M}$ *and given by*

$$J(\hat{\mathbf{v}}) = \left\|\hat{\mathbf{T}}\hat{\mathbf{v}} + \hat{\mathbf{C}}\hat{\mathbf{N}}\right\|_{\mathcal{P}}^2 - \gamma^2\|\hat{\mathbf{v}}\|_{\mathcal{M}}^2 \tag{15}$$

(a) $\|\hat{\mathbf{T}}\| < \gamma$ *if and only if there exists* $\beta>0$*, such that*

$$\|\hat{\mathbf{T}}\hat{\mathbf{v}}\|_{\mathcal{P}}^2 - \gamma^2\|\hat{\mathbf{v}}\|_{\mathcal{M}}^2 \le -\beta\|\hat{\mathbf{v}}\|_{\mathcal{M}}^2 \qquad \forall\hat{\mathbf{v}} \in \mathcal{M} \tag{16}$$

Consequently, if $\|\hat{\mathbf{T}}\| < \gamma$, *then* (15) *always achieves a unique finite maximum over* $\mathcal{M}$.

(b) If $\|\hat{\mathbf{T}}\| > \gamma$ *then* (15) *does not achieve a finite maximum over* $\mathcal{M}$*, i.e.* $\sup_{\hat{\mathbf{v}}\in\mathcal{M}} J(\hat{\mathbf{v}}) = +\infty$.

It mean that $\|\hat{\mathbf{T}}\| = \inf\gamma$ over all γ such that the maximization of (15) has a finite solution. The required value of γ can be found with arbitrary accuracy, e.g. by means of the bisection method. Equivalence between the maximization of the functional (15) and the existence of a solution to the corresponding Riccati difference equations can be exploited.

Estimation of the operator norm using the method of parameterised functional minimization in general can takes large computational power.

2.2. Running finite time horizon based norm estimation

In order to make computationally efficient norm estimation, following approach based of finite-time horizon norm is proposed.

Definition 1. Amplification energy factor k_e for system with zero initial condition $\mathbf{x}_0=\mathbf{0}$ is given in following way

$$k_e = \frac{\|\hat{\mathbf{y}}\|}{\|\hat{\mathbf{v}}\|} = \sqrt{\frac{\hat{\mathbf{y}}^{\mathrm{T}}\hat{\mathbf{y}}}{\hat{\mathbf{v}}^{\mathrm{T}}\hat{\mathbf{v}}}} = \sqrt{\frac{\sum_{i=1}^{N} y^2(i)}{\sum_{i=1}^{N} v^2(i)}} \tag{17}$$

For systems unstable in the input-output sense output energy grows unboundedly for bounded input signals, i.e. $\sup_{\hat{\mathbf{v}}\neq 0}(k_e)=\infty$. It implies infinite value of the norm of transfer operator, i.e.

$$\|\hat{\mathbf{T}}\| = \sup_{\hat{\mathbf{v}}\neq 0}(k_e) \tag{18}$$

where the norm $\|\hat{\mathbf{T}}\| \to \infty$.

For systems stable in the input-output sense output energy is bonded for bounded input signals, i.e. $0 \le k_e < \infty$. It implies finite value of the norm of transfer operator $\|\hat{\mathbf{T}}\|$.

Let us assume that a system defined on infinite time horizon will be considered as a system defined on finite time horizon with length N. The norm of transfer operator of the system defined on finite time horizon N be denoted in following way:

$$\|\hat{\mathbf{T}}_{[N]}\| \tag{19}$$

where

$$\underset{N\in\mathbb{Z}}{\forall} \quad \|\hat{\mathbf{T}}_{[N-1]}\| \le \|\hat{\mathbf{T}}_{[N]}\| \tag{20}$$

If the norm of transfer operator defined on infinite time horizon is finite $\|\hat{\mathbf{T}}\| = c$ then there exist a limit c such that:

$$\lim_{N\to\infty} \|\hat{\mathbf{T}}_{[N]}\| = c \tag{21}$$

Thus for large enough lengths of the time horizon it may be concluded that finite time horizon norm is an approximation of the infinite time horizon norm, i.e.:

$$\underset{N\ge N_0}{\forall} \|\hat{\mathbf{T}}_{[N]}\| \cong \|\hat{\mathbf{T}}\| \tag{22}$$

Relative approximation error can be expressed by following equation:

$$\delta(\hat{\mathbf{T}}, N) = \left| \frac{\|\hat{\mathbf{T}}_{[N]}\|}{\|\hat{\mathbf{T}}\|} - 1 \right| \tag{23}$$

Although it is impossible to find simple relation between the relative error δ and the length of the time horizon N for general time-varying system $\hat{\mathbf{T}}$, we show that the method is relatively simple and efficient for discrete-time, time-varying systems norm estimation.

3. NUMERICAL ANALYSIS FOR PERIODIC TIME-VARYING SYSTEM

The system under consideration is special case of the linear time-varying system whereas $\mathbf{A}(k)$ is the time-varying system matrix with invariant eigenvalues. The system is characterized by constant (time-invariant) eigenvalues of the system matrix despite changes in its entries. This idea is borrowed from De La Sen (2002), Khalil (1996). The additional parameter ε allows changes of the system with a degree of non-stationarity as well as the pole location. Eigenvalues of matrix $\mathbf{A}(k)$ are inside the unitary circle, but can be either stable or unstable with respect to switching in the structure of the system. The deciding factor is the switching interval defined by the parameter ε. System matrices (1)-(2) are the following:

$$\mathbf{A}(k) = \mathbf{A}_\kappa, \quad \mathbf{B}(k) = \begin{bmatrix}1 & 0\end{bmatrix}^{\mathrm{T}}, \quad \mathbf{C}(k) = \begin{bmatrix}0 & 1\end{bmatrix}, \quad \mathbf{D}(k) = 0 \tag{24}$$

where

$$\begin{aligned} &\mathbf{A}_0 = \begin{bmatrix} 2 & 1.2 \\ -2 & -1 \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} -1 & -2 \\ 1.2 & 2 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} -1 & 1.2 \\ -2 & 2 \end{bmatrix}, \\ &\mathbf{A}_3 = \begin{bmatrix} 2 & -2 \\ 1.2 & -1 \end{bmatrix}, \ \kappa = \mathrm{floor}\left(\mathrm{rem}\left(\frac{k}{\varepsilon}, 4 \right) \right) \end{aligned} \tag{25}$$

Variable κ denotes rounding towards negative infinity (floor) of the remanent (signed remainder of k/ε after division by 4). Eigenvalues of the matrix $\mathbf{A}(k)$ are independent of the parameter ε and equal to $0.5 \pm 0.3873\mathrm{i}$ for all k.

In fact value of the parameter ε significantly changes properties of the system. Small values $\varepsilon<2.8$ implies unstable character of the system whereas large values results in stable, switching system. Figure 1 shows values of the transfer operator norm $\|\hat{\mathbf{T}}_{[N]}\|$ vs. length of the time horizon N for $\varepsilon = 5$. Value estimated using lifting techniques is equal to $\|\hat{\mathbf{T}}\| = 12.9849$ and depicted by dotted line. As can be seen from fig. 1 estimated norm fast reach neighbourhood of the real value. It takes only about 27 time steps.

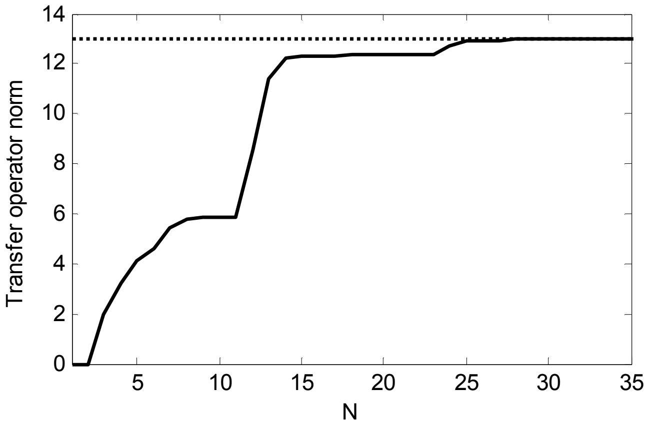

Figure 1. Norm of transfer operator for finite time horizon discrete switching system (24)-(25) with $\varepsilon = 5$ vs. the length of the time horizon *N*.

Relative error for the same system computed for the length of the time horizons up to 500 is depicted in fig. 2. From practical point of view relative error for norm estimation below 10^{-2} is in most cases sufficient, in this case it takes only 27 time steps what is relatively fast, even for second order system but with variability period of $4\varepsilon = 20$ time steps.

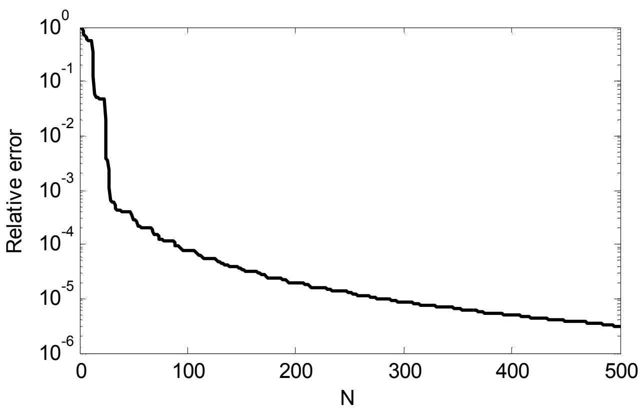

Figure 2. Relative error of the transfer operator norm computed on finite time horizon for discrete switching system (24)-(25) with $\varepsilon = 5$ vs. the length of the time horizon *N*.

4. CONCLUSION

In the paper a novel approach for the estimation of the operator norm is proposed. Particularly infinite dimensional transfer operator norm of dynamical discrete-time, periodical time-varying stable systems can be estimated using block matrix operator notation for transfer operator defined on finite time horizon. The minimal length of the time horizon required for computations is dependent both on the dominant time constant of the system and the variability period of the system matrices. Open problems are connected mostly with estimating the minimal length of the time horizon required for computations. Further investigations should concern extending the method for wider class of the time-varying systems e.g. for other common classes, i.e. almost periodic systems etc.

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