University of Szczecin

Institute of Mathematics

Mateusz Firkowski

Selected problems of stability and observability of Timoshenko beams

Supervisor: Prof. Grigory Sklyar
Co-supervisor: Dr. Jarosław Woźniak

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Preface

Differential equations first came into existence with the invention of calculus in seventeenth century by I. Newton and G. W. Leibniz. Then, this theory was investigated by J. Bernoulli, J. Riccati, A. Clairaut, J.-B. R. d'Alembert, L. Euler, D. Bernoulli, and J.-L. Lagrange. In the early stages of development, differential equations were mainly used to describe physical phenomena such as string vibration or heat flow. Nowadays, the theory of differential equations is widely used not only in mathematics but also in related fields such as physics (e.g. Schrödinger equation) or chemistry (e.g. reaction rates), and even unrelated fields such as biology (e.g. Lotka-Volterra model), economics (e.g. dynamics of gross domestic product) or medicine (e.g. SIR model used in mathematical modeling of infectious diseases). That is why its development is so important for all science.

There are several types of classification of differential equations. The most common divisions in the literature are based on a distinction whether the equation is ordinary or partial, linear or nonlinear, homogeneous or nonhomogeneous. We will briefly discuss the first division. An ordinary differential equation is an equation involving an unknown function of one independent variable and its derivatives. They are used to describe objects in which the state is an element of the finite-dimensional space. Their theory is well known, in most cases their solutions can be expressed in terms of integrals. A partial differential equation is an equation involving an unknown function of two or more independent variables and certain of its partial derivatives. They are widely used to describe phenomena in nature such as sound, vibrations or fluid flow. In this case, the state is described by a function, that is, an element of an infinite-dimensional space. As opposed to ordinary differential equations, in most cases it is rather impossible to write an exact solution of a partial differential equation. In this case, we can try to estimate the behavior of the solutions. Such problems are part of an emerging field of mathematics, namely the mathematical control theory. The mathematical control theory is an area of mathematics dealing with the analysis and modeling of objects and processes, treated as dynamical systems with control. Thanks to it, we can study such properties of those systems as stability (small perturbations of initial conditions lead to small perturbations of solutions), controllability (the existence of input that carries out the system in a finite period of time to a given state under the fulfillment of initial conditions) or observability (one can determine the behavior of the entire system from the knowledge of the system's outputs).

The following dissertation is devoted to the analysis of stability and observability of a particular model of vibrations in beams, the so-called Timoshenko beam model. This model is described by a set of linear partial differential equations.

Analysis of stability of complex systems described by partial differential equations is an important problem of mathematical control theory. Every complex physical system is composed of simple components, e.g. beams or plates. Thus, the stability of beams has been the subject of several investigation during the last two decades. The majority of publications concentrated on Euler-Bernoulli beam model, e.g. [7, 9, 38, 62]. Various stability problems were in the scope of considerations in those papers. The problem of two identical Euler-Bernoulli beams coupled end to end via an energy-dissipating joint was considered in [7]. R. Curtain and K. Morris proved L^2-stability of the system after including damping operator in a clamped-free Euler beam with shear force control model in [9]. J. Valverde and D. García-Vallejo [62] observed additional effects of Coriolis forces, and they investigated their influence on stability of the beam rotating with a critical angular velocity. The stability analysis of a system composed of rotating beams on a flexible, circular fixed ring, using Routh-Hurwitz criterion is presented by N. Lesaffre, J.-J. Sinou and F. Thouverez [38].

Timoshenko beam model is a generalization of Euler beam model, taking into account additional rotation of a cross-section area. Again, many authors considered different stability aspects for this generalized object. J. U. Kim and Y. Renardy [27] showed that the Timoshenko beam can be uniformly stabilized by means of a boundary control. A. Manevich and Z. Kołakowski [41] studied the dynamics of Timoshenko beam made of a viscoelastic material. A. Zuyev and O. Sawodny [74] consider stabilizing observer of a system describing the motion of a flexible-link manipulator with a payload under the action of gravity. M. I. Mustafa and S. A. Messaoudi [42] considered a Timoshenko system with viscoelastic boundary conditions localized on a part of the boundary. A. Guesmia and S. A. Messaoudi [18] considered a one-dimension Timoshenko system with different speeds of wave propagation and with only one control given by a viscoelastic term on the angular rotation equation. Z. J. Han and G. Q. Xu [22] studied the stabilization problem and Riesz basis property of two serially connected Timoshenko beams. M. Gugat [19] studied the problem to move the beam from a given initial state to a position of rest, where the movement is controlled by the angular acceleration of the axis to which the beam is clamped.

Many authors studied the problem of a damping in dynamical systems. M. A. Shubov in [50] developed spectral and asymptotic analysis for a class of non-self-adjoint operators which are the dynamics generators for the systems governed by the equations of the spatially nonhomogeneous Timoshenko beam model with a 2-parameter family of dissipative boundary conditions. In [69], G. Q. Xu and S. P. Yung studied the exponential decay rate of a Timoshenko beam system with boundary damping. J. E. M. Rivera and A. I. Ávila [46] considered the uniform stabilization of a hybrid elastic model consisting of a Timoshenko beam and a tip load at the free end of the beam. W. He and S. S. Ge presented the modeling and vibration control problem of a satellite with two flexible solar panels in [23]. In [24] the same authors considered the vibration control design for an Euler-Bernoulli beam with the boundary output constraint. M. Bassam, D. Mercier, S. Nicaise and A. Wehbe in [4] studied the indirect boundary stabilization of the Timoshenko system with only one dissipation law.

More complicated models, including not only vibrations of the beam, were also studied in last decades. Since 1999 W. Krabs and G. M. Sklyar considered different controllability and stabilizability aspects of a special, undamped model of a rotating Timoshenko beam clamped to the motor disk in [32–35]. In [32], the problem of transferring the beam from a position of rest into another given position of rest within a given time was solved. They showed in [33], how to choose a feedback control allowing to stabilize the system in a preassigned position of a rest. W. Krabs, G. M. Sklyar and J. Woźniak obtained conditions of exact controllability under the assumption that the physical parameter γ appearing in the model equation is rational in [35].

Here two main problems are investigated. Firstly, stability of a particular model of vibrations of Timoshenko beams with a weak (distributed) damping connected to rotations of cross-sections of the beam, of deflections of the center line of the beam, and of both is analyzed. In one of the cases considered, for some values of physical parameters of the beam the optimal stability margin phenomenon may be observed, which means that under some conditions there exists an optimal value of a damping coefficient, that is a coefficient that guarantees the fastest possible decay of norms of solutions of the system. Secondly, exact observability problem of vibration of undamped Timoshenko beam is studied.

Controllability, its dual notion of observability, and related problems were widely investigated in last few decades. The cases of approximate and spectral controllability and the corresponding dual notions of observability were one of main directions (see the book by D. Salamon [48] and references therein). B. Jacob and H. Zwart in [26] showed that infinite-dimensional version of the Hautus test is sufficient for exact observability of certain exponentially stable systems generated by C_0-group. They also proved that this Hautus test is in general not sufficent for approximate observability of strongly stable systems even if the system is modeled by a contraction semigroup and the observation operator is bounded. T. Duyckaerts, X. Zhang and E. Zuazua in [11] proved the optimality of the observability inequality for parabolic systems with potentials in even space dimensions $n \geq 2$. G. O. Antunes, F. D. Araruna and A. Mercado [1] considered the dynamical one-dimensional Mindlin-Timoshenko system for beams. They obtained a global exact controllability result for this semilinear system with superlinear nonlinearities. For this purpose, they established an observability estimate for the linearized system with bounded potentials. Moreover, they obtained an explicit estimate of the observability constant in terms of the norms of potentials. A. Sengouga in [49] studied the wave equation in an interval with two linearly moving endpoints and establishes observability results, at one or at both endpoints, in a sharp time. J. H. Chen [8] considered infinite-time exact observability of Volterra systems in Hilbert spaces. He established sufficient conditions under which infinite-time exact observability of a Volterra system follows from that of the corresponding Cauchy system without convolution term. B. H. Haak and D.-T. Hoang in [20] investigated admissibility and exact observability estimates of boundary observation and interior point observation of a 1-dimensional wave equation on a time-dependent domain for sufficiently regular boundary functions. They also discussed moving observers inside the noncylindrical domain and simultaneous observability results. S. Cai and M. Xiao [6] studied the boundary observability for one-dimensional wave equation associated with nonlinear boundary condition that can generate complex dynamics. They discussed the exact observability and approximate observability, respectively, in terms of three different types of common boundary observations by studying the wave interactions on the boundary directly. W. Zhang, W. X. Zheng and B.-S. Chen [73] studied detectability, observability and related Lyapunov-type theorems of linear discrete-time time-varying stochastic systems with multiplicative noise.

Before proceeding with observability considerations for Timoshenko beam model, some important general observability results are proven. Those conclusions can be used in many practical systems and devices in various domains of science and technology (e.g. observability of vibrational processes in automation and robotics). As an example of the practical use of the obtained results, the problem of exact observability of vibrating Timoshenko beam model is considered. As opposed to string and Euler-Bernoulli beam models, the eigensystem of this model is not a Riesz basis, only the system with divided differences forms Riesz basis for a sufficiently large time T [32]. From the mathematical point of view the problems can be reduced to solvability of certain trigonometric non-Fourier moment problems with two asymptotically close families of exponentials. Conditions for solvability of those problems were first proposed in terms of convergence of series of divided differences of moments by D. Ullrich [61]. It turned out that in some situations the conditions of convergence can be understood as a kind of smoothness of projections of end-states of the system to some special subspaces [53].

The dissertation is divided into a preface and four chapters.

In the first chapter, we discuss the elements of classical operator theory which are important later in the work. At the beginning we focus on linear operators. Then we introduce strongly continuous groups and semigroups associated with operators of the differential equations. Next, we extend the concept of eigenvalues and eigenvectors in infinite-dimensional spaces, i.e. we present basics of spectral theory. At the end, we state theorems about generating strongly continuous groups and semigroups and we present basic definitions and properties of Riesz basis and Riesz-spectral operators.

The second chapter is devoted to introducing differential equations describing the vibrations of Timoshenko beam models considered in the following chapters.

In the third chapter, we analyze stability of Timoshenko beam model including damping effects. To this end, we carry out spectral analysis of the operators associated with differential equations describing the system under consideration. Then we prove that in some particular cases those operators satisfy spectrum determined growth condition, which means that the location of the spectrum allows us to determine the stability margin of the system. Furthermore, we investigate the existence of an optimal decay rate. At the end we compare the obtained results with other damping operators.

In the fourth chapter, we consider the problem of exact observability of a general class of distributed parameter systems in Hilbert spaces. We prove that the system with some specific assumptions on spectrum and eigensystem is not exactly observable in default topology setting. Then we find stronger topology for state observation for which the system becomes exactly observable. In the end, we show that clamped-free Timoshenko beam system satisfies obtained results.

Contents

Chapter 1

Elements of Classical Operator Theory

This chapter is devoted to introduce basic definitions and theorems which are necessary in the main part of the dissertation. We start with a general theory of linear operators. Next, we discuss the properties of strongly continuous groups and semigroups and their association with differential equations. Then, we introduce spectral theory and generation theorems. At the end of this chapter we formulate some results on Riesz bases and define an important class of operators, i.e. Riesz spectral-operators.

The results discussed here are well known in functional analysis and may be found in books on this subject or monographs in control theory. Contents of this chapter are based on the following references: [10] for Linear Operators, [10, 14, 72] for Strongly Continuous Groups and Semigroups, [10, 14] for Spectral Theory, [10, 14, 39, 43, 72] for Generation Theorems and [10, 16, 61, 70, 75] for Riesz Basis and Riesz-spectral Operators.

1.1 Linear Operators

In this section we focus on transformation T from one normed linear space X to another Y. We assume that X and Y will be either Banach or Hilbert spaces. Later in this section we discuss basic properties of linear operators.

At the beginning, we start with the definition of a linear operator.

Definition 1.1. A *linear operator*, or simply an *operator*, T from linear space X to a linear space Y over the same field $\mathcal{F}$ is a map $T : D(T) \subset X \to Y$, such that $D(T)$ is a subspace of X, and for all $x_1, x_2 \in D(t)$ and scalars α, it holds that:

i) $T(x_1 + x_2) = T(x_1) + T(x_2)$,

ii) $T(\alpha x) = \alpha T(x)$.

The set $D(T)$ in above definition is called the *domain* of the operator T. Importantly, the same mapping defined in different domains gives us different operators. For example, consider the operator $T_1 : D(T_1) \to L^2(0,1)$ defined by $T_1 x = 2x$ with the domain $D(T_1) = \{x \in L^2(0,1) \mid x \text{ continuous}\}$ and the operator $T_2 : D(T_2) \to L^2(0,1)$ defined by $T_2 x = 2x$ with the domain $D(T_2) = L^2(0,1)$. Naturally, the operator T_1 differs from the operator T_2.

Definition 1.2. The set of all possible images of the operator $T : D(T) \to Y$ is a subspace of Y, in general. It is called the *range* of T and we denote this by $\operatorname{ran} T$. If the range of an operator is finite-dimensional, then we say that the operator has *finite rank*.

Now we turn to the inverse operators.

Definition 1.3. An operator $T : D(T) \subset X \to Y$ between two linear spaces X and Y is *invertible* if there exists a map $S : D(S) := \operatorname{ran} T \subset Y \to X$ such that:

i) $STx = x$, $x \in D(T)$,

ii) $TSy = y$, $y \in \operatorname{ran} T$.

S is called the *algebraic inverse* of T and we write $T^{-1} = S$.

Lemma 1.4 (see [10])**.** *Linear operators T from X to Y, where X and Y are linear vector spaces, have the following properties:*

a) T is invertible if and only if T is injective*, that is, $Tx = 0$ implies $x = 0$.*

b) If T is an operator and it is invertible, then its algebraic inverse is also linear.

The set of all elements in the domain of T such that $Tx = 0$ is called the *kernel* of T and is denoted by $\ker T$. If T is a linear operator, then $\ker T$ is a linear subspace. From the above lemma we see that the linear operator T has an inverse if $\ker T = \{0\}$.

Now we proceed with notions of continuous and bounded operators.

Definition 1.5. A map $F : D(F) \subset X \to Y$ between two normed linear spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is said to be *continuous at* $x_0 \in X$ if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|F(x) - F(x_0)\|_Y < \varepsilon$, whenever $\|x - x_0\|_X < \delta$. F is *continuous on* $D(F)$ if it is continuous at every point in $D(F)$.

Definition 1.6. Let T be a linear operator from $D(T) \subset X \to Y$, where X and Y are normed linear spaces. T is a *bounded linear operator* or T is *bounded* if there exists a real number c such that for all $x \in D(T)$

$$\|Tx\|_Y \leq c\|x\|_X.$$

Definition 1.7. Let T be a bounded linear operator from $D(T) \subset X$ to Y. We define its *norm*, $\|T\|$, by

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X}.$$

An equivalent definition of $\|T\|$ is

$$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\|_X = 1}} \|Tx\|_Y.$$

The relation between continuous and bounded linear operators are given in the following theorem.

Theorem 1.8 (see [10])**.** *If $T : D(T) \subset X \to Y$ is a linear operator, where X and Y are normed linear spaces, then:*

a) T is continuous if and only if T is bounded.

b) If T is continuous at a single point, it is continuous on $D(T)$.

Now we define a space of bounded linear operators.

Definition 1.9. If X and Y are normed linear spaces, we define the normed linear space $\mathcal{L}(X, Y)$ to be the space of bounded linear operators from X to Y with $D(T) = X$ and with norm given by Definition 1.7.

For the special case that $X = Y$ we denote $\mathcal{L}(X, X)$ by $\mathcal{L}(X)$.

Lemma 1.10 (see [10])**.** *Let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators from X to Y. Then the following properties hold:*

a) If Y is a Banach space, then so is $\mathcal{L}(X, Y)$.

b) If X, Y and Z are normed linear spaces, $T_1 \in \mathcal{L}(X, Y)$ and $T_2 \in \mathcal{L}(Y, Z)$, then T_3, defined by $T_3x = T_2(T_1x)$, is an element of $\mathcal{L}(X, Z)$ and $\|T_3\| \leq \|T_1\|\|T_2\|$.

c) $\mathcal{L}(X)$ *is an algebra; that is* αT_1*,* T_1+T_2 *and* T_1T_2 *are in* $\mathcal{L}(X)$ *for every* T_1, T_2 *in* $\mathcal{L}(X)$*; furthermore,* $\|T_1T_2\| \leq \|T_1\|\|T_2\|$.

The concept of convergence in the space of the bounded linear operators is as follows.

Definition 1.11. Let $\{T_n, n \geq 1\}$ be a sequence of bounded linear operators in $\mathcal{L}(X,Y)$, where X and Y are normed linear spaces, then

i) T_n *converges uniformly* to T, if

$$\|T_n - T\|_{\mathcal{L}(X,Y)} \to 0 \text{ as } n \to \infty,$$

ii) T_n *converges strongly* to T, if

$$\|T_n x - Tx\|_Y \to 0 \text{ as } n \to \infty \text{ for all } x \in X.$$

In the case of the bounded linear operators dependent on a parameter t, where t is from some interval in $\mathbb{R}$, we can define strong and uniform continuity with respect to t in an analogous manner.

Definition 1.12. If $T(t)$ is in $\mathcal{L}(X,Y)$ for every $t \in [a,b]$, where X and Y are normed linear spaces, then

i) $T(t)$ is *uniformly continuous at* t_0, if

$$\|T(t) - T(t_0)\|_{\mathcal{L}(X,Y)} \to 0 \text{ as } t \to t_0,$$

ii) $T(t)$ is *strongly continuous at* t_0, if

$$\|T(t)x - T(t_0)x\|_Y \to 0 \text{ for all } x \in X \text{ as } t \to t_0.$$

Another subclass of bounded linear operators with useful properties are compact operators.

Definition 1.13. Let X and Y be normed linear spaces. An operator $T \in \mathcal{L}(X,Y)$ is said to be a *compact operator* if T maps bounded sets of X onto relatively compact sets of Y. An equivalent definition is that T is linear and for any bounded sequence $\{x_n\}$ in X, $\{Tx_n\}$ has a convergent subsequence in Y.

Some properties of compact operators are summarized in the following lemma.

Lemma 1.14 (see [10]). *Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear operator. Then the following assertions hold:*

a) If T is bounded and $\dim(T(X)) < \infty$, then the operator T is compact.

b) If $\dim(X) < \infty$, then the operator T is compact.

c) The range of T is separable if T is compact.

d) If S, U are elements of $\mathcal{L}(X_1, X)$ and $\mathcal{L}(Y, Y_1)$, respectively, and $T \in \mathcal{L}(X, Y)$ is compact, then so is UTS.

e) If $\{T_n\}$ is a sequence of compact operators from X to the Banach space Y, that converge uniformly to T, then T is a compact operator.

f) The identity operator, I, on the Banach space X is compact if and only if $\dim(X) < \infty$.

g) If T is a compact operator in $\mathcal{L}(X, Y)$ whose range is a closed subspace of Y, then the range of T is finite-dimensional.

Integral operators are an important example of compact operators on the space $L^2(a, b)$.

Theorem 1.15 (see [10]). *Let $k(t, s)$ be an element of $L^2([a, b] \times [a, b])$. Then the operator K from $L^2(a, b)$ to $L^2(a, b)$ defined by*

$$(Ku)(t) = \int_a^b k(t, s)u(s)ds$$

is a compact operator.

In this section we considered mainly on bounded linear operators. However, in applications we will often find *unbounded* linear operators.

Definition 1.16. Let X and Y be normed linear spaces and $T : D(T) \subset X \to Y$ a linear operator. The *graph* $\mathcal{G}(T)$ is the set

$$\mathcal{G}(T) = \{(x, Tx) \mid x \in D(T)\}$$

in the product space $X \times Y$.

Definition 1.17. A linear operator T is said to be *closed* if its graph $\mathcal{G}(T)$ is a closed linear subspace of $X \times Y$. Alternatively, T is closed if whenever

$$x_n \in D(T),\ n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} Tx_n = y,$$

it follows that $x \in D(T)$ and $Tx = y$.

It is rather difficult to prove from the above definition that the operator is closed. The next theorem gives us a criterion for checking the closedness of the operators.

Theorem 1.18 (see [10]). *Assume that X and Y are Banach spaces and let T be a linear operator with domain $D(T) \subset X$ and range Y. If, in addition, T is invertible with $T^{-1} \in \mathcal{L}(Y, X)$ then T is a closed linear operator.*

Similarly to the above theorem, we can introduce a condition for check the boundedness of the linear operators.

Theorem 1.19 (**Closed Graph Theorem**, see [10]). *A closed linear operator defined on all of a Banach space X into a Banach space Y is bounded.*

A Hilbert space is a special case of a Banach space with a norm induced by an inner product. Thus, all the properties discussed previously in this section are true also for the Hilbert space. However, a Hilbert space gives us additional properties of the operators which we briefly discuss below.

Theorem 1.20 (**Riesz Representation Theorem**, see [10]). *If H is a Hilbert space, then every element h in H induces a bounded linear functional f defined by*

$$f(x) = \langle x, h \rangle_H.$$

On the other hand, for every bounded linear functional f on H, there exists a unique vector $h_0 \in H$ such that

$$f(x) = \langle x, h_0 \rangle_H \text{ for all } x \in H,$$

and furthermore, $\|f\| = \|h_0\|$.

Note that the last equality the norm on the left-hand side is the norm in the space of linear functionals, and the norm on the right-hand side is the norm in the vector space H. The consequence of the Riesz Representation Theorem is the existence of the adjoint operator.

Definition 1.21. Let $T \in \mathcal{L}(H_1, H_2)$, where H_1 and H_2 are Hilbert spaces. Then there exists a unique operator $T^* \in \mathcal{L}(H_2, H_1)$ that satisfies

$$\langle Tx_1.x_2 \rangle_{H_2} = \langle x_1, T^* x_2 \rangle_{H_1} \text{ for all } x_1 \in H_1, x_2 \in H_2.$$

The operator T^* is called the *adjoint operator* of T.

Some properties of adjoint operators are given in the following lemma.

Lemma 1.22 (see [10])**.** *Let $T_1, T_2 \in \mathcal{L}(H_1, H_2)$ and $S \in \mathcal{L}(H_2, H_3)$, where H_1, H_2 and H_3 are Hilbert spaces. The adjoint has the following properties:*

a) $I^* = I$.

b) $(\alpha T_1)^* = \overline{\alpha} T_1^*$.

c) $\|T_1^*\| = \|T_1\|$.

d) $(T_1 + T_2)^* = T_1^* + T_2^*$.

e) $(ST_1)^* = T_1^* S^*$.

f) $\|T_1^* T_1\| = \|T_1\|^2$.

We can also define the adjoint of an unbounded linear operator.

Definition 1.23. Let A be a linear operator on a Hilbert space H. Assume that the domain of A, $D(A)$, is dense in H. Then the *adjoint operator* $A^* : D(A^*) \subset H \to H$ of A is defined as follows. The domain $D(A^*)$ of A^* consists of all $y \in H$ such that there exists a $y^* \in H$ satisfying

$$\langle Ax, y \rangle = \langle x, y^* \rangle \text{ for all } x \in D(A).$$

For each such $y \in D(A^*)$ the *adjoint operator* A^* is then defined in terms of y^* by

$$A^* y = y^*.$$

It can be shown that if A is a closed, densely defined operator, then $D(A^*)$ is dense in H and A^* is closed. In the following lemma, we have some results for adjoint operators.

Lemma 1.24 (see [10])**.** *Let A be an arbitrary, densely defined operator and let T be a bounded linear operator defined on the whole of the Hilbert space H. The following holds:*

a) $(\alpha A)^* = \overline{\alpha} A^*$, $D((\alpha A)^*) = D(A^*)$ *if* $\alpha \neq 0$ *and* H *if* $\alpha = 0$.

b) $(A + T)^* = A^* + T^*$, *with domain* $D((A + T)^*) = D(A^*)$.

c) If A has a bounded inverse, i.e., there exists an $A^{-1} \in \mathcal{L}(H)$ such that $AA^{-1} = I_H$, $A^{-1}A = I_{D(A)}$, then A^* also has a bounded inverse and $(A^*)^{-1} = (A^{-1})^*$.

Now we define a special class of operators.

Definition 1.25. We say that a densely defined, linear operator A is *symmetric* if for all $x, y \in D(A)$

$$\langle Ax, y \rangle = \langle x, Ay \rangle.$$

A symmetric operator is *self-adjoint* if $D(A^*) = D(A)$.

For a self-adjoint operator A, we have that $\langle Ax, x \rangle = \langle x, Ax \rangle$, which means that $\langle Ax, x \rangle$ is real for all $x \in D(A)$. If this value is additionally nonnegative, we can introduce special name of these operators.

Definition 1.26. A self-adjoint operator A on the Hilbert space H is *nonnegative* if

$$\langle Ax, x \rangle \geq 0 \text{ for all } x \in D(A),$$

A is *positive* if

$$\langle Ax, x \rangle > 0 \text{ for all nonzero } x \in D(A),$$

and A is *coercive* if there exists and $\varepsilon > 0$ such that

$$\langle Ax, x \rangle \geq \varepsilon \|x\|^2 \text{ for all } x \in D(A).$$

We shall use the notation $A \geq 0$ for nonnegativity of the self-adjoint operator A, and $A > 0$ for positivity. Furthermore, if T, S are self-adjoint operators in $\mathcal{L}(H)$, then we shall write $T \geq S$ for $T - S \geq 0$.

For self-adjoint and nonegative operator we are able to find *square root*.

Lemma 1.27 (see [10])**.** *If A is self-adjoint and nonnegative, then A has a unique nonnegative* square root $A^{\frac{1}{2}}$*, so that $D\left(A^{\frac{1}{2}}\right) \supset D(A)$, $A^{\frac{1}{2}}x \in D\left(A^{\frac{1}{2}}\right)$ for all $x \in D(A)$, and $A^{\frac{1}{2}}A^{\frac{1}{2}}x = Ax$ for $x \in D(A)$. Furthermore, if A is positive, then $A^{\frac{1}{2}}$ is positive too.*

Another important class of operators are *skew-adjoint* operators.

Definition 1.28. Operator A is a *skew-adjoint* operator, if $D(A^*) = D(A)$ and $A^* = -A$.

Analogously to self-adjoint operators, for every skew-adjoint operator A the range of $\langle Ax, x \rangle$ is purely imaginary.

1.2 Strongly Continuous Groups and Semigroups

The ordinary differential equations of the form

$$\dot{z}(t) = Az(t) + Bu(t),\ z(0) = z_0,\ t \geq 0,$$

are often used to describe control systems with a finite-dimensional state space. The solutions of these equations are given by the formula

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds,\ t \geq 0, \tag{1.1}$$

where

$$T(t) = e^{At},\ t \geq 0,$$

is a solution of the equation

$$\dot{\varphi}(t) = A\varphi(t),\ \varphi(0) = I.$$

Note that (1.1) is a well defined integral in the sense of Bochner (see [10]).

However, there exist a large number of systems which cannot be represented by a finite number of parameters. Examples of such systems are string vibrations or heat flow in a rod. To describe their state, we will use a function that is an element of an infinite-dimensional function space. Control theory of infinite dimensional systems becomes significantly complicated. The situation is similar to passing from ordinary to partial differential equations. This motivates the necessity for generalizing the concept of a fundamental solution and introducing semigroup theory.

Let H be a separable complex Hilbert space. Here and below $\mathbb{R}^+$ will denote the set of nonnegative real numbers.

Definition 1.29. A *strongly continuous semigroup* (or C_0-*semigroup*) is an operator-valued function $T(t)$ from $\mathbb{R}^+$ to $\mathcal{L}(H)$ that satisfies the following properties:

i) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$,

ii) $T(0) = I$,

iii) $\|T(t)x - x\| \to 0$ as $t \to 0^+$ $\forall x \in H$.

Remark 1.30. If these properties hold for $\mathbb{R}$ instead of $\mathbb{R}^+$, we call $T(t)$ a *strongly continuous group* (or C_0-*group*) on H.

In the work we will write alternatively a strongly continuous semigroup, C_0-semigroup or even semigroup. The following theorem gives us some basic properties of semigroups.

Theorem 1.31 (see [10])**.** *A strongly continuous semigroup $T(t)$ on a Hilbert space H has the following properties:*

a) $\|T(t)\|$ *is bounded on every finite subinterval of* $[0, \infty)$*,*

b) $T(t)$ *is strongly continuous for all* $t \in [0, \infty)$*,*

c) For all $x \in H$ *we have that* $\frac{1}{t}\int\limits_0^t T(s)x ds \to x$ *as* $t \to 0^+$*,*

d) If $\omega_0(T) := \inf\limits_{t>0}\left(\frac{1}{t}\log\|T(t)\|\right)$*, then* $\omega_0(T) = \lim\limits_{t\to\infty}\left(\frac{1}{t}\log\|T(t)\|\right) < \infty$*,*

e) $\forall\omega > \omega_0(T)$ *there exists a constant* M_ω *such that* $\forall t \geq 0$*,*

$$\|T(t)\| \leq M_\omega e^{\omega t}. \tag{1.2}$$

Definition 1.32. The constant $\omega_0(T)$ defined in the above theorem is called *growth bound* of the semigroup. If this does not lead to a misunderstanding, we will write shortly ω_0 instead of $\omega_0(T)$. Moreover, a semigroup is called *bounded* if for $\omega = 0$ and some M_0 in inequality (1.2) is fulfilled, and *contractive* if for $\omega_0 = 0$ and $M_0 = 1$ in inequality (1.2) is valid.

In the case of a finite-dimensional space, the operator A is connected with the fundamental solution e^{At} by

$$\left.\left(\frac{d}{dt}e^{At}\right)\right|_{t=0} = A.$$

Now we associate in a similar way an unbounded operator A to a C_0-semigroup $T(t)$.

Definition 1.33. The *infinitesimal generator* A of a C_0-semigroup on a Hilbert space H is defined by

$$Ax = \lim_{t\to 0^+}\frac{1}{t}(T(t) - I)x, \tag{1.3}$$

whenever the limit exists; the domain of A, $D(A)$, being the set of elements in H for which the limit exists.

Remark 1.34. A is the *infinitesimal generator* of a C_0-group if t in limit in (1.3) tends to 0 instead of 0^+.

Example 1.35. Let A be a $n \times n$ real or complex matrix. Then, the family

$$T(t) = e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \tag{1.4}$$

is a semigroup with generator A. Formula (1.4) is also true if we assume that A is a bounded linear operator on H.

In general the above definition allows us to calculate the generator of a C_0-semigroup, but it is rather difficult to apply. Some basic properties of a generator of a C_0-semigroup are given in the following theorem.

Theorem 1.36 (see [10])**.** *Let $T(t)$ be a strongly continuous semigroup on a Hilbert space H with infinitesimal generator A. Then the following hold:*

a) For $x \in D(A)$, $T(t)x \in D(A)$ $\forall t \geq 0$,

b) $\frac{d}{dt}(T(t)x) = AT(t)x = T(t)Ax$ for $x \in D(A)$, $t > 0$,

c) $\frac{d^n}{dt^n}(T(t)x) = A^n T(t)x = T(t)A^n x$ for $x \in D(A^n)$, $t > 0$,

d) $T(t)x - x = \int_0^t T(s)Axds$ for $x \in D(A)$,

e) $\int_0^t T(s)xds \in D(A)$ and $A\int_0^t T(s)xds = T(t)x - x$ for all $x \in H$, and $D(A)$ is dense in H,

f) A is a closed linear operator,

g) $\bigcap_{n=1}^{\infty} D(A^n)$ is dense in H.

1.3 Spectral Theory

Spectral theory deals with the generalization of the notion of eigenvalues and eigenvectors in infinite dimensional spaces. For this purpose, we will consider abstract equation of the form

$$(\lambda I - A)x = y,$$

where A is a closed linear operator on a complex Banach space X with $D(A) \subset X$, $x, y \in X$, and $\lambda \in \mathbb{C}$. Our main task is asking under what conditions $(\lambda I - A)$ has a bounded inverse on the particular Banach space X.

We begin our considerations with the introduction of a resolvent set.

Definition 1.37. Let A be a closed linear operator on a (complex) normed linear space X. We say that λ is in the *resolvent set* $\rho(A)$ of A, if $(\lambda I - A)^{-1}$ exists and is a bounded linear operator on a dense domain of X.

It can be shown that $\lambda \in \rho(A)$ if and only if $(\lambda I - A)^{-1} \in \mathcal{L}(X)$. We shall call $(\lambda I - A)^{-1}$ the *resolvent operator* of A and denote as $R(\lambda, A)$. The *spectrum* of the operator is complement of the resolvent set and is decomposed into three disjoint sets as defined below.

Definition 1.38. Let A be a closed linear operator on a (complex) normed linear space X. The *spectrum* of A is defined to be

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

The *point spectrum* is

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is not injective}\}.$$

The *continuous spectrum* is

$$\begin{aligned}\sigma_c(A) &= \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is injective}, \overline{\operatorname{ran}(\lambda I - A)} = X, \text{ but} \\ &\qquad (\lambda I - A)^{-1} \text{ is unbounded}\} \\ &= \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is injective}, \overline{\operatorname{ran}(\lambda I - A)} = X, \text{ but} \\ &\qquad \operatorname{ran}(\lambda I - A) \neq X\}.\end{aligned}$$

The *residual spectrum* is

$$\sigma_r(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is injective, but } \operatorname{ran}(\lambda I - A) \text{ is not dense in } X\}.$$

So $\sigma(A) = \sigma_p(A) \cup \sigma_p(A) \cup \sigma_r(A)$ (cf. [10]).

A point $\lambda \in \sigma_p(A)$ is an *eigenvalue*, and $x \neq 0$ such that $(\lambda I - A)x = 0$, an *eigenvector*.

Definition 1.39. Let λ_0 be an eigenvalue of the closed linear operator A on the Banach space X. Suppose further that this eigenvalue is *isolated*, that is there exists an open neighbourhood O of λ_0 such that $\sigma(A) \cap O = \{\lambda_0\}$. We say that λ_0 has *order* ν_0 if for every $x \in X$

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu_0} (\lambda I - A)^{-1} x$$

exists, but there exists an x_0 such that the following limit does not

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu_0 - 1} (\lambda I - A)^{-1} x_0.$$

If for every $\nu \in \mathbb{N}$ there exists an $x_\nu \in X$ such that the limit

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu_0} (\lambda I - A)^{-1} x_\nu$$

does not exist, then the order of λ_0 is infinity.

For the isolated eigenvalue λ_0 of finite order ν_0, its (*algebraic*) *multiplicity* is defined as $\dim(\ker(\lambda_0 I - A)^{\nu_0})$. The elements of $\ker(\lambda_0 - A)^{\nu_0}$ are called the *generalized eigenvectors* corresponding to λ_0.

Now we give a theorem in which we define the formula for the *integral representation of the resolvent*. The formula relates semigroups to the resolvents of their generators.

Theorem 1.40 (see [14]). *Let $T(t)$ be a strongly continuous semigroup on the Banach space X and take constants $\omega \in \mathbb{R}$, $M \geq 1$ (see Theorem 1.31.e)) such that*

$$\|T(t)\| \leq M e^{\omega t}$$

for $t \geq 0$. For the generator A of $T(t)$ the following properties hold.

i) If $\lambda \in \mathbb{C}$ is such that $R(\lambda, A)x = \int\limits_0^\infty e^{-\lambda s} T(s) x ds$ exists for all $x \in X$, then $\lambda \in \rho(A)$.

ii) If $\operatorname{Re} \lambda > \omega$, then $\lambda \in \rho(A)$, and the resolvent is given by the integral expression in i).

iii) $\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re} \lambda - \omega}$ for all $\operatorname{Re} \lambda > \omega$.

Property ii) in the above theorem means that the spectrum of a generator of a semigroup is always contained in a left half–plane. The number determining the smallest such half–plane is an important characteristic of any linear operator and is defined as follows.

Definition 1.41. To any linear operator A we associate its *spectral bound* defined by

$$s(A) = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}.$$

As a consequence of Theorem 1.40.ii) the following relation between the growth bound of a strongly continuous semigroup and the spectral bound of its generator holds.

Corollary 1.42. For a strongly continuous semigroup $T(t)$ with generator A, one has

$$-\infty \leq s(A) \leq \omega_0 < +\infty.$$

Of course, the natural question seems to be—when $s(A) = \omega_0$? We will answer this question in Section 1.5.

For our considerations, *operators with compact resolvent* are a very important class of operators. An operator with compact resolvent has a pointwise spectrum.

Lemma 1.43 (see [14])**.** *Let A be a closed linear operator with $0 \in \rho(A)$ and A^{-1} compact. The spectrum of A consists of only isolated eigenvalues with finite multiplicity.*

The next lemma is about the form of the spectrum of self-adjoint operators.

Lemma 1.44 (see [10])**.** *If A is a self-adjoint operator on the Hilbert space H, then $\sigma(A) \subset \mathbb{R}$.*

Corollary 1.45. Analogously to Lemma 1.44, if A is a skew-adjoint operator on the Hilbert space H, then $\sigma(A) \subset i\mathbb{R}$.

1.4 Generation Theorems

As we mentioned in Section 1.2 for any given semigroup we are able to find its generator. The converse is more interesting. Does any operator generate semigroup? Theorems which answer this question are called *generation theorems*. Thus, we now present the characterization of generators of arbitrary strongly continuous semigroup.

We start with a fundamental result of Hille-Yosida theorem.

Theorem 1.46 (**Hille-Yosida Theorem**, see [10])**.** *A necessary and sufficient condition for a closed, densely defined, linear operator on a Hilbert space H[1] to be the infinitesimal generator of a C_0-semigroup is that there exist real numbers M, ω, such that for all real $\alpha > \omega$, $\alpha \in \rho(A)$, the resolvent set of A, and*

$$\|R(\alpha, A)^n\| \leq \frac{M}{(\alpha - \omega)^n} \text{ for all } n \geq 1,$$

where $R(\alpha, A) = (\alpha I - A)^{-1}$ is the resolvent operator. In this case

$$\|T(t)\| \leq Me^{\omega t}.$$

[1]Theorem is also true in the case of Banach space X, see e.g. [14]

As a general rule, operator A is a generator when the spectrum $\sigma(A)$ lies in some left half-plane and growth estimates of the form

$$\|R(\alpha, A)^n\| \leq \frac{M}{(\alpha-\omega)^n}$$

hold for all powers of the resolvent $R(\lambda, A)$ in some right half-plane (or on some semiaxis (ω, ∞)). The condition with estimation of the norm of the resolvent is rather complicated and difficult to check in general situation. Simpler conditions for an operator to generate a contraction semigroup are given in Lumer-Phillips theorem, but before we present them, we need to define the *dissipative operator*.

Definition 1.47. A linear operator A with $D(A)$ contained in a Banach space X is called *dissipative* if

$$\|(\lambda - A)x\| \geq \lambda \|x\|$$

for all $\lambda > 0$ and $x \in D(A)$.

Remark 1.48 (see [14])**.** For operators in a Hilbert space there is an equivalent condition for checking the dissipativity of the operator, namely

$$\mathrm{Re}\langle Ax, x\rangle \leq 0$$

for all $x \in D(A)$.

Theorem 1.49 (**Lumer-Phillips Theorem**, see [39])**.** *A necessary and sufficient condition for a linear operator A with a dense domain in a Banach space X to generate a strongly continuous semigroup of contraction is that A be dissipative and that* $\mathrm{ran}(I - A) = X$.

Another important result in applications is the fact that the sum of a generator and a bounded linear operator is a generator.

Theorem 1.50 (**Phillips Theorem**, see [43,72])**.** *If an operator A generates a semigroup on a Banach space X and $K : X \to X$ is a bounded linear operator then the operator $A + K$ with the domain identical to $D(A)$ is also a generator.*

In general, semigroups are defined only for $t \geq 0$. Now we can extend the theory of generating strongly continuous semigroups to strongly continuous groups defined for all $t \in \mathbb{R}$. In order to do that we present the following lemmas. At the beginning, we define $T^+(t) := T(t)$ and $T^-(t) = T(-t)$ for $t \geq 0$.

Lemma 1.51 (see [10]). *If $T(t)$ is a C_0-group, then $T^+(t)$ and $T^-(t)$ are C_0-semigroups.*

Lemma 1.52 (see [10]). *Let A be the infinitesimal generator of the C_0-semigroup $T^+(t)$, then $-A$ is the infinitesimal generator of the C_0-semigroup $T^-(t)$,*

Lemma 1.53 (see [10]). *A is the infinitesimal generator of the C_0-group if and only if A is the infinitesimal generator of the C_0-semigroup and $-A$ is the infinitesimal generator of the C_0-semigroup.*

Lemma 1.54 (see [10]). *Let A be an infinitesimal generator of a C_0-group. The spectrum of A lies in a strip along the imaginary axis, i.e., $\sigma(A) \subset \{z \in \mathbb{C} \mid |\operatorname{Re}(s)| < \beta\}$ for some $\beta > 0$.*

1.5 Riesz Basis and Riesz-spectral Operators

One of the most important type of basis in a Hilbert space is the orthonormal basis. Another very useful class of bases are the bases equivalent to the orthonormal basis, the so-called Riesz basis. They are helpful in the case where the operators encountered have eigenvectors that are not orthonormal but form the Riesz basis.

We begin with the following definition.

Definition 1.55. A sequence $\{\phi_j\}_{j=1}^{\infty}$ of vectors of a Banach space X (or Hilbert space) is called a *Schauder basis* of this space if every vector $x \in X$ can be expanded in a unique way in a series

$$x = \sum_{j=1}^{\infty} c_j \phi_j$$

which converges in the norm of the space X.

The coefficients c_j in a series expansion of a vector x in the case of Hilbert space can be determined by

$$c_j = \langle x, \psi_j \rangle, \ j = 1, 2, \ldots$$

where $\{\psi_j\}_{j=1}^{\infty}$ is a *biorthogonal sequence* corresponding to $\{\phi_j\}_{=1}^{\infty}$ and is defined as follows.

Definition 1.56. Let $\{\phi_j\}_{j=1}^{\infty}$ and $\{\psi_j\}_{j=1}^{\infty}$ be sequences of elements of the Hilbert space H. Then $\{\psi_j\}_{j=1}^{\infty}$ is *biorthogonal* to $\{\phi_j\}_{j=1}^{\infty}$ if for $i, j \in \mathbb{N}$

$$\langle \phi_i, \psi_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The sequence $\{\phi_j\}_{j=1}^{\infty}$ is independent if there exists a sequence $\{\psi_j\}_{j=1}^{\infty}$ which is biorthogonal to $\{\phi_j\}_{j=1}^{\infty}$. The sequence $\{\phi_j\}_{j=1}^{\infty}$ is complete in H if 0 is the only element of H which is orthogonal to each ϕ_j ($j \in \mathbb{N}$). Equivalently, $\{\phi_j\}_{j=1}^{\infty}$ is complete if and only if its linear span is dense in H.

Riesz basis is a generalization of orthonormal basis.

Definition 1.57. A basis for a Hilbert space is a *Riesz basis* if it is equivalent to an orthonormal basis, that is, if it is obtained from an orthonormal basis by means of a bounded invertible operator.

The next theorem gives a number of characteristic properties of Riesz bases.

Theorem 1.58 (see [16, 70])**.** *The following assertions are equivalent.*

a) The sequence $\{\phi_j\}_{j=1}^{\infty}$ forms a Riesz basis for H.

b) There is an equivalent[2] inner product on H, with respect to which the sequence $\{\phi_j\}_{j=1}^{\infty}$ becomes an orthonormal basis for H.

c) The sequence $\{\phi_j\}_{j=1}^{\infty}$ is complete in H, and there exist positive constants A and B such that for an arbitrary positive integer n and arbitrary scalars $c_1, \ldots, c_n$ one has

$$A \sum_{i=1}^{n} |c_i|^2 \leq \left\| \sum_{i=1}^{n} c_i \phi_i \right\|^2 \leq B \sum_{i=1}^{n} |c_i|^2.$$

d) The sequence $\{\phi_j\}_{j=1}^{\infty}$ is complete in H, and its Gram matrix

$$[\langle \phi_i, \phi_j \rangle]_{i,j=1}^{\infty}$$

generates a bounded invertible operator on ℓ^2.

e) The sequence $\{\phi_j\}_{j=1}^{\infty}$ is complete in H and possesses a complete biorthogonal sequence $\{\psi_j\}_{j=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} |\langle x, \phi_i \rangle| < \infty \text{ and } \sum_{i=1}^{\infty} |\langle x, \psi_i \rangle| < \infty$$

for every $x \in H$.

[2]Two inner products are said to be *equivalent* if they generate equivalent norms.

Remark 1.59. The inequality from property c) in the above theorem is also true for infinite number of elements, i.e.,

$$A\sum_{i=1}^{\infty}|c_i|^2 \leq \left\|\sum_{i=1}^{\infty} c_i\phi_i\right\|^2 \leq B\sum_{i=1}^{\infty}|c_i|^2.$$

Another very useful result for generating Riesz basis from eigenvectors of the infinitesimal generator of a strongly continuous group gives us Zwart theorem.

Theorem 1.60 (**Zwart Theorem**, see [75])**.** *Let A be the infinitesimal generator of C_0-group $T(t)$ on the Hilbert space H. We denote the eigenvalues of A by λ_n (counting with multiplicity), and the corresponding (normalized) eigenvectors by $\{\phi_n\}$. If the following two conditions hold,*

a) The span of the eigenvectors form a dense set in H.

b) The point spectrum has a uniform gap*, i.e.,*

$$\inf_{n\neq m}|\lambda_n - \lambda_m| > 0,$$

then the eigenvectors form a Riesz basis on H.

Now we define a class of operators with simple eigenvalues whose eigenvectors formed a Riesz basis.

Definition 1.61. Suppose that A is a linear, closed operator on a Hilbert space H, with simple eigenvalues $\{\lambda_n, n \geq 1\}$ and suppose that the corresponding eigenvectors $\{\phi_n, n \geq 1\}$ form a Riesz basis in H. If the closure of $\{\lambda_n, n \geq 1\}$ is totally disconnected, then we call A a *Riesz-spectral operator*.

By *totally disconnected* we mean that no two points $\lambda, \mu \in \overline{\{\lambda_n, n \geq 1\}}$ can be joined by a segment lying entirely in $\overline{\{\lambda_n, n \geq 1\}}$.

Theorem 1.62 (see [10])**.** *Suppose that A is a Riesz-spectral operator with simple eigenvalues $\{\lambda_n, n \geq 1\}$ and corresponding eigenvectors $\{\phi_n, n \geq 1\}$. Let $\{\psi_n, n \geq 1\}$ be the eigenvectors of A^* such that $\langle\phi_n, \psi_m\rangle = \delta_{nm}$. Then A satisfies:*

a) $\rho(A) = \{\lambda \in \mathbb{C} \mid \inf_{n\geq 1}|\lambda - \lambda_n| > 0\}$, $\sigma(A) = \overline{\{\lambda_n, n \geq 1\}}$, and for $\lambda \in \rho(A)$ the inverse $(\lambda I - A)^{-1}$ is given by

$$(\lambda I - A)^{-1} = \sum_{n=1}^{\infty}\frac{1}{\lambda - \lambda_n}\langle\cdot, \psi_n\rangle\phi_n.$$

b) A has the representation

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \psi_n \rangle \phi_n$$

for $x \in D(A)$, and

$$D(A) = \{x \in H \mid \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, \psi_n \rangle|^2 < \infty\}.$$

c) A is the infinitesimal generator of a C_0-semigroup if and only if $\sup\limits_{n \geq 1} \mathrm{Re}(\lambda_n) < \infty$ and $T(t)$ is given by

$$T(t) = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle \cdot, \psi_n \rangle \phi_n.$$

d) The growth bound of the semigroup $T(t)$ is given by

$$\omega_0 = \inf_{t>0} \left(\frac{1}{t} \log \|T(t)\| \right) = \sup_{n \geq 1} \mathrm{Re}(\lambda_n).$$

As we mentioned in Section 1.3 spectral bound of generator A is always less or equal to the growth bound of a strongly continuous semigroup $T(t)$ generated by A (see Corollary 1.42). From property d) in Theorem 1.60 we have that for Riesz-spectral operator spectral bound is equal to growth bound, i.e. $\omega_0(T) = s(A)$. In this case we will say that Riesz-spectral operator satisfies *spectrum determined growth condition*.

Chapter 2

Timoshenko Beam Theory

There are four widely used models for describing vibrations in beams. These four theories are Euler-Bernoulli, shear, Rayleigh and Timoshenko [13, 21]. The Euler-Bernoulli beam theory, sometimes called classical beam theory, was presented in the 40s of the 18th century [59]. This model includes the strain energy due to the bending and the kinetic energy due to the lateral displacement and is widely used because of simplicity and sufficiently good approximations for many engineering problems. Other theories have introduced additional physical effects to improve the classical beam theory. The shear model adds shear deformation effects and was for the first time introduced by W. J. M. Rankine in 1858 [45]. At a similar time (in 1877) Lord Rayleigh [55] extended Euler-Bernoulli equation by incorporating the rotary inertia of the cross-section. Note that taking into account the effects of the moment of inertia was proposed 18 years earlier by J. A. C. Bresse in [5]. Finally, in 1921 S. P. Timoshenko published his best known paper [58] (cited about 1500 times) in which he presented generalization of Euler-Bernoulli equation with above-mentioned physical effects, i.e. shear and rotary inertia effects. There are some historical controversies concerning this date [12], because this theory can be found in the earlier works of Timoshenko, in the book on elasticity in the Russian language from 1916 [56], and for the first time in English language in Croatian journal from 1920 [57].

From the mathematical point of view, the main difference between Timoshenko and the rest of beam models (Euler-Bernoulli, shear, Rayleigh) is that the operator connected with the system of partial differential equations has a spectrum that consists of not one eigenvalues family but two families of eigenvalues. Furthermore, the eigensystem of Timoshenko beam model is not a Riesz basis, only the system with divided differences forms a Riesz basis for a sufficiently large time T. This is one of the main reasons that this system is very interesting for analyzing different problems from mathematical control theory.

2.1 Modeling Rotating Timoshenko Beam

Now, we will introduce a detailed derivation of the vibration equations of the rotating Timoshenko beam, which can be found originally in the monograph of W. Krabs and G. M. Sklyar [34].

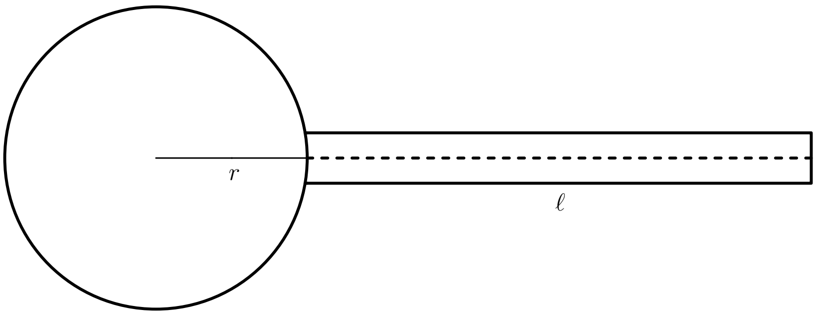

Figure 2.1: Beam at the position of rest

We consider the rotation of a two dimensional beam in a horizontal plane whose left end is clamped into the disk of a driving motor (see Fig. 2.1). Let $r > 0$ be the radius of the disk and let $\theta = \theta(t)$ be the rotation angle as a function of the time $t \geqslant 0$. Let $\ell > 0$ be the length of the beam and for every $x \in [0, \ell]$ and let $\omega(x)$ be the cross-section of the beam at x which is assumed to be an interval which is symmetric to $y = 0$ (see Fig. 2.2). This implies that in the case of a rigid rotation of the beam (i.e. without additional vibration) the position of an arbitrary point $(x, y) \in [0, \ell] \times \omega(x)$ under the rotation angle θ is given by

$$\vec{R}_\theta(x, y) = (r + x)\vec{e}_1(\theta) + y\vec{e}_2(\theta),$$

where

$$\vec{e}_1(\theta) = (\cos\theta, \sin\theta) \text{ and } \vec{e}_2(\theta) = (-\sin\theta, \cos\theta).$$

Now let $\vec{r}(x, y)$ be the additional displacement of (x, y) under the influence of vibration. For small displacements we can assume that

$$\vec{r}(x, y) = \vec{r}(x, 0) + y\vec{r}_y(x, 0),\ x \in [0, \ell],\ y \in \omega(x).$$

Let

$$\vec{r}(x, 0) = \tilde{w}_1(x, \theta)\vec{e}_1(\theta) + \tilde{w}_2(x, \theta)\vec{e}_2(\theta)$$

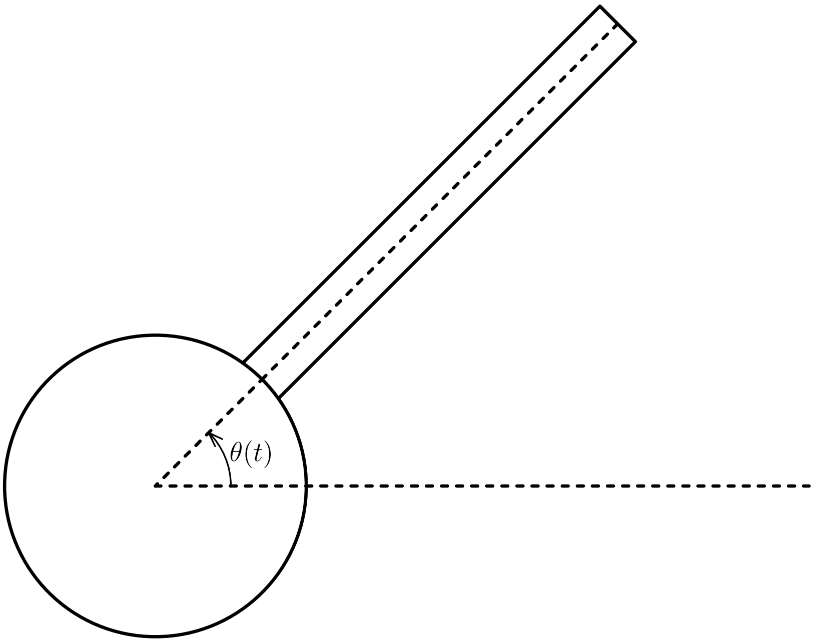

Figure 2.2: Rigid beam in motion

for $x \in [0, \ell]$. Furthermore, assume that

$$\vec{r}_y(x, 0) = \tilde{\xi}(x, \theta)\vec{e}_1(\theta)$$

for $x \in [0, \ell]$. The displacement of the point $(x, y) \in [0, \ell] \times \omega(x)$ under the influence of rotation and additional vibration is then given by

$$\begin{aligned}\tilde{\vec{R}}(x, y, \theta) &= \vec{R}_\theta(x, y) + \vec{r}(x, 0) \\ &= \Big(r + x + \tilde{w}_1(x, \theta) + \tilde{\xi}(x, \theta)y\Big)\,\vec{e}_1(\theta) + (y + \tilde{w}_2(x, \theta))\,\vec{e}_2(\theta).\end{aligned}$$

for $x \in [0, \ell]$ and $y \in \omega(x)$. Let us put

$$\begin{aligned}\vec{R}(x, y, t) &= \tilde{\vec{R}}(x, y, \theta(t)), \\ w_1(x, t) &= \tilde{w}_1(x, \theta(t)), \\ w_2(x, t) &= \tilde{w}_2(x, \theta(t)), \\ \xi(x, t) &= \tilde{\xi}(x, \theta(t)).\end{aligned}$$

Then it follows that

$$\vec{R}(x, y, t) = (r + x + w_1(x, t) + \xi(x, t)y)\,\vec{e}_1\,(\theta(t)) + (y + w_2(x, t))\,\vec{e}_2\,(\theta(t))$$

for $x \in [0,\ell]$, $y \in \omega(x)$, $t \in \mathbb{R}^+$. Let us denote the derivative with respect to t by a dot. Then

$$\begin{aligned}
\dot{\vec{R}}(x,y,t) &= \Big(\dot{w}_1(x,t) + \dot{\xi}(x,t)y\Big)\,\vec{e}_1\left(\theta(t)\right) \\
&\qquad + \left(r + x + w_1(x,t) + \xi(x,t)y\right)\dot{\theta}(t)\vec{e}_2\left(\theta(t)\right) \\
&\qquad + \dot{w}_2(x,t)\vec{e}_2\left(\theta(t)\right) - \left(y + w_2(x,t)\right)\dot{\theta}(t)\vec{e}_1\left(\theta(t)\right) \\
&= \Big(\dot{w}_1(x,t) + \dot{\xi}(x,t)y - \left(y + w_2(x,t)\right)\dot{\theta}(t)\Big)\,\vec{e}_1\left(\theta(t)\right) \\
&\qquad + \Big(\left(r + x + w_1(x,t) + \xi(x,t)y\right)\dot{\theta}(t) + \dot{w}_2(x,t)\Big)\,\vec{e}_2\left(\theta(t)\right).
\end{aligned}$$

This implies

$$\begin{aligned}
\left\|\dot{\vec{R}}(x,y,t)\right\|^2 &= \Big(\dot{w}_1(x,t) + \dot{\xi}(x,t)y - \left(y + w_2(x,t)\right)\dot{\theta}(t)\Big)^2 \\
&\qquad + \Big(\left(r + x + w_1(x,t) + \xi(x,t)y\right)\dot{\theta}(t) + \dot{w}_2(x,t)\Big)^2
\end{aligned}$$

for $x \in [0,\ell]$, $y \in \omega(x)$, $t \in \mathbb{R}^+$. Assume additionally that

$$w_1 \cdot \dot{\theta} = w_2 \cdot \dot{\theta} = \xi \;\; \cdot \dot{\theta} = 0.$$

Then

$$\left\|\dot{\vec{R}}(x,y,t)\right\|^2 = \Big(\dot{w}_1(x,t) + \dot{\xi}(x,t)y - \dot{\theta}(t)y\Big)^2 + \Big(\left(r + x\right)\dot{\theta}(t) + \dot{w}_2(x,t)\Big)^2$$

for $x \in [0,\ell]$, $y \in \omega(x)$, $t \in \mathbb{R}^+$.

Let $\tilde{\rho}$ be the (constant) density of the material of the beam. Then, for every $t \in \mathbb{R}^+$, the kinetic energy of the beam is given by

$$K(w_1,w_2,\xi)(t) = \frac{\tilde{\rho}}{2}\int_0^\ell \int_{\omega(x)} \left\|\dot{\vec{R}}(x,y,t)\right\|^2 dydx.$$

Let, for every $x \in [0,\ell]$, $A(x)$ and $M(x)$ denote cross-section area and area moment of inertia, respectively, then

$$A(x) = \int_{\omega(x)} dy \text{ and } M(x) = \int_{\omega(x)} y^2 dy.$$

Then it follows that

$$\begin{aligned}
K(w_1,w_2,\xi)(t) = \frac{\tilde{\rho}}{2}\Bigg(&\int_0^\ell A(x)\dot{w}_1(x,t)^2 + M(x)\left(\dot{\xi}(x,t)^2 + \dot{\theta}(t)^2\right) \\
&- 2M(x)\dot{\xi}(x,t)\dot{\theta}(t)dx + \int_0^\ell A(x)(r+x)^2\dot{\theta}(t)^2 \\
&+ A(x)\dot{w}_2(x,t)^2 + 2A(x)(r+x)\dot{\theta}(t)\dot{w}_2(x,t)dx\Bigg)
\end{aligned}$$

for $t \in \mathbb{R}^+$, hence

$$K(w_1, w_2, \xi)(t) = \frac{\tilde{\rho}}{2} \int_0^\ell A(x)\dot{w}_1(x,t)^2 + M(x)\left(\dot{\xi}(x,t) - \dot{\theta}(t)\right)^2 + A(x)\left((r+x)\dot{\theta}(t) + \dot{w}_2(x,t)\right)^2 dx$$

for $t \in \mathbb{R}^+$.

In [68] it is shown that, for every $t \in \mathbb{R}^+$, the potential energy of the beam is given by

$$U(w_1, w_2, \xi)(t) = \frac{1}{2} \int_0^\ell E(x)\left(M(x)\xi'(x,t)^2 + A(x)w_1'(x,t)^2\right) + K(x)\left(w_2'(x,t) + \xi(x,t)\right)^2 dx,$$

where $E(x)$ is Young's modulus, $K(x)$ is the shear modulus, and "$\,'\,$" denotes the derivative with respect to x.

For $x = 0$, we have the requirement

$$\vec{R}(0, y, t) = r\vec{e}_1(\theta) + y\vec{e}_2(\theta)$$

for all $y \in \omega(x)$ and $t \in \mathbb{R}^+$. This implies

$$w_1(0,t) = w_2(0,t) = \xi(0,t) = 0 \text{ for all } t \in \mathbb{R}^+.$$

Now let $T > 0$ be chosen arbitrarily and let V_T be the subspace of $C^2([0,\ell] \times [0,T], \mathbb{R}^3)$ consisting of all vector functions $(w_1, w_2, \xi) \in C^2([0,\ell] \times [0,T], \mathbb{R}^3)$ which satisfy the boundary conditions

$$w_1(0,t) = w_2(0,t) = \xi(0,t) = 0 \text{ for all } t \in [0,T].$$

In order to get a vector function $(w_1, w_2, \xi) \in V_T$ which describes the motion of the beam we have to minimize the Lagrange functional

$$L(w_1, w_2, \xi) = \int_0^T K(w_1, w_2, \xi)(t) - U(w_1, w_2, \xi)(t)dt$$

on V_T. A necessary condition for $(w_1, w_2, \xi) \in V_T$ to minimize $L = L(w_1, w_2, \xi)$ on V_T is

$$D_{(w_1,w_2,\xi)}L(h_1, h_2, h_3) = 0 \text{ for all } (h_1, h_2, h_3) \in V_T,$$

where $D_wL(h)$ denotes the Gateaux derivative of L at the $w \in V_T$ in the direction $h \in V_T$. This is equivalent to the statement that

$$
\begin{aligned}
&\int_0^T \tilde{\rho} \int_0^\ell A(x)\dot{w}_1\dot{h}_1 + M(x)\dot{\xi}\dot{h}_3 - M(x)\dot{\theta}\dot{h}_3 + A(x)\dot{w}_2\dot{h}_2 + A(x)(r+x)\dot{\theta}\dot{h}_2 dx \\
&- \int_0^\ell E(x)\left(M(x)\xi' h_3' + A(x)w_1' h_1'\right) \\
&+ K(x)\left(w_2' h_2' + \xi h_3 + \xi h_2' + w_2' h_3\right) dxdt \\
= 0&
\end{aligned}
$$

for all $(h_1, h_2, h_3) \in V_T$.

Now let us assume that the beam is uniform, that is

$$E(x) = E,\ K(x) = K \text{ and } \omega(x) = \omega \text{ for all } x \in [0, \ell]$$

which implies

$$A(x) = A = \int_\omega dy \text{ and } M(x) = M = \int_\omega y^2 dy \text{ for all } x \in [0, \ell].$$

Then it follows that

$$
\begin{aligned}
&\int_0^T \tilde{\rho} \int_0^\ell -A\ddot{w}_1 h_1 - M\ddot{\xi} h_3 + M\ddot{\theta} h_3 - A\ddot{w}_2 h_2 - A(r+x)\ddot{\theta} h_2 dx \\
&+ \int_0^\ell E\left(M\xi'' h_3 + Aw_1'' h_1\right) + K\left(w_2'' h_2 - \xi h_3 + \xi' h_2 - w_2' h_3\right) dxdt \\
&- \int_0^T E\left(M\xi'(\ell,t)h_3(\ell,t) + Aw_1'(\ell,t)h_1(\ell,t)\right) \\
&+ K\left(w_2'(\ell,t)h_2(\ell,t) + \xi(\ell,t)h_2(\ell,t)\right) dt \\
=& \int_0^T \int_0^\ell \left(-\tilde{\rho} A\ddot{w}_1 + EAw_1''\right) h_1 dx + EAw_1'(\ell,t)h_1(\ell,t) dt \\
&+ \int_0^T \int_0^\ell \left(-\tilde{\rho} A\ddot{w}_2 - \tilde{\rho} A(r+x)\ddot{\theta} + K\left(w_2'' + \xi'\right)\right) h_2 dx \\
&+ K\left(w_2'(\ell,t) + \xi(\ell,t)\right) h_2(\ell,t) dt \\
&+ \int_0^T \int_0^\ell \left(-\tilde{\rho} M\left(\ddot{\xi} - \ddot{\theta}\right) + EM\xi'' - K\left(w_2' + \xi\right)\right) h_3 dx
\end{aligned}
$$

$$
\begin{aligned}
&\quad - EM\xi'(\ell,t)h_3(\ell,t)dt \\
&= 0
\end{aligned}
$$

for all $(h_1, h_2, h_3) \in V_T$ with $h_i(x,0) = h_i(x,T) = 0$, $i = 1,2,3$, for all $x \in [0,\ell]$. This implies the differential equations

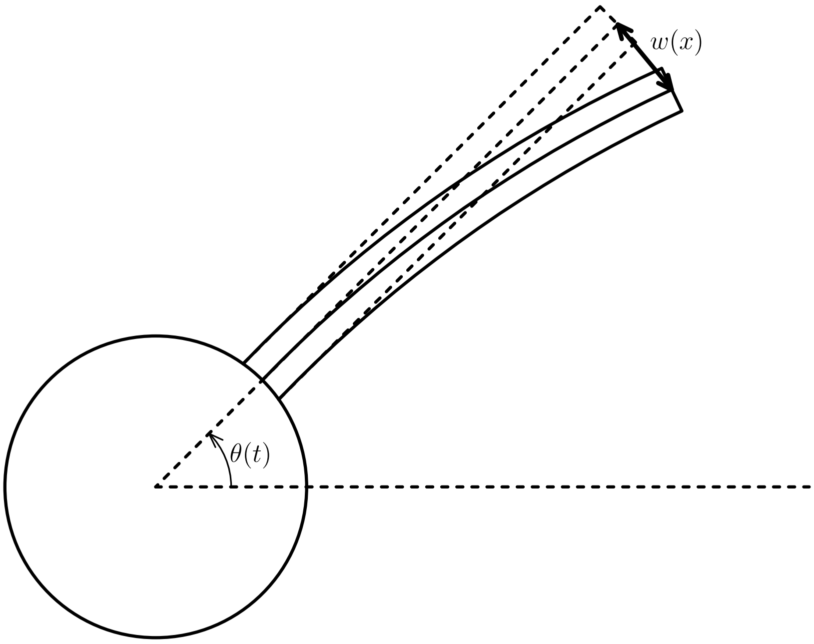

Figure 2.3: Deflection of the center line of the rotating beam

$$
\begin{cases}
\ddot{w}_1(x,t) - \dfrac{EA}{\rho} w_1''(x,t) = 0, \\
\ddot{w}_2(x,t) - \dfrac{K}{\rho}\left(w_2''(x,t) + \xi'(x,t)\right) = -\ddot{\theta}(t)(r+x), \\
\ddot{\xi}(x,t) - \dfrac{EA}{\rho}\xi''(x,t) + \dfrac{K}{I}\left(w_2'(x,t) + \xi(x,t)\right) = \ddot{\theta}(t)
\end{cases}
$$

for $x \in (0,\ell)$ and $t \in (0,T)$ with $\rho = \tilde{\rho}A$, $I = \tilde{\rho}M$, where ρ is linear density, I denotes moment of inertia, and the boundary conditions

$$
w_1'(\ell,t) = w_2'(\ell,t) + \xi(\ell,t) = \xi'(\ell,t) = 0 \text{ for } t \in [0,T].
$$

If we prescribe initial conditions in the form

$$
\begin{aligned}
&w_1(x,0) = \dot{w}_1(x,0) = 0,\ w_2(x,0) = \dot{w}_2(x,0) = 0, \\
&\xi(x,0) = \dot{\xi}(x,0) \text{ for all } x \in [0,\ell],
\end{aligned}
$$

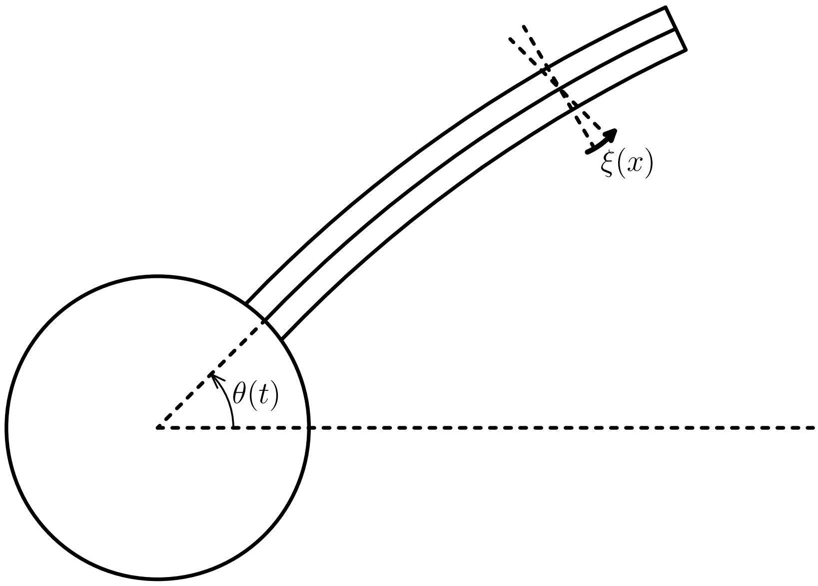

Figure 2.4: The rotation angle of the cross-section area of the rotating beam

then it follows that $w_1 \equiv 0$ on $[0, \ell] \times [0, T]$.

Finally, assuming $w = w_2$, equations of vibration of rotating Timoshenko beam (see Fig. 2.3 and 2.4) takes form

$$\begin{cases} \ddot{w}(x,t) - \dfrac{K}{\rho}\left(w''(x,t) + \xi'(x,t)\right) = -\ddot{\theta}(t)(r+x), \\ \ddot{\xi}(x,t) - \dfrac{EA}{\rho}\xi''(x,t) + \dfrac{K}{I}\left(w'(x,t) + \xi(x,t)\right) = \ddot{\theta}(t) \end{cases} \tag{2.1}$$

for $x \in (0, \ell)$ and $t > 0$, with boundary conditions

$$\begin{cases} w(0,t) = \xi(0,t) = 0, \\ w'(\ell,t) + \xi(\ell,t) = 0, \\ \xi'(\ell,t) = 0 \end{cases}$$

for $t > 0$.

2.2 Operator Equation of Undamped Beam

Following [34], we want to rewrite equations of motion of rotating Timoshenko beam (2.1) in the operator equation form

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \; t \in (0, T). \tag{2.2}$$

At the beginning, we normalize the units to simplify the system (2.1). To this end, we will introduce an appropriate change of variables

$$\tilde{x} = \sqrt{\frac{\rho}{I}}x,\ \tilde{t} = \sqrt{\frac{K}{I}}t,$$

of constants

$$\tilde{\ell} = \sqrt{\frac{\rho}{I}}\ell,\ \tilde{T} = \sqrt{\frac{K}{I}}T,\ \tilde{r} = \sqrt{\frac{\rho}{I}}r,$$

and of functions

$$\tilde{w}(\tilde{x},\tilde{t}) = \sqrt{\frac{\rho}{I}}w(x,t),\ \tilde{\xi}(\tilde{x},\tilde{t}) = \xi(x,t),\ \tilde{\theta}(\tilde{t}) = \theta(t)$$

for $x \in [0,\ell]$ and $t \in [0,T]$. Then, system (2.1) can be transferred into

$$\begin{cases} \ddot{\tilde{w}}(\tilde{x},\tilde{t}) - \tilde{w}''(\tilde{x},\tilde{t}) - \tilde{\xi}'(\tilde{x},\tilde{t}) = -\ddot{\tilde{\theta}}(\tilde{t})(\tilde{r}+\tilde{x}), \\ \ddot{\tilde{\xi}}(\tilde{x},\tilde{t}) - \dfrac{EA}{K}\tilde{\xi}''(\tilde{x},\tilde{t}) + \tilde{w}'(\tilde{x},\tilde{t}) + \tilde{\xi}(\tilde{x},\tilde{t}) = \ddot{\tilde{\theta}}(\tilde{t}) \end{cases}$$

in $(0,\tilde{\ell}) \times (0,\tilde{T})$. Finally, after putting $\tilde{\ell} = 1$, denoting $\frac{EA}{K} = \gamma^2$ and replacing "$\tilde{}$" by " ", $u(t) := \ddot{\theta}(t)$, we obtain free-dimensional two partial differential equations of the form

$$\begin{cases} \ddot{w}(x,t) - w''(x,t) - \xi'(x,t) = -u(t)(r+x), \\ \ddot{\xi}(x,t) - \gamma^2\xi''(x,t) + w'(x,t) + \xi(x,t) = u(t) \end{cases} \tag{2.3}$$

for $x \in (0,1)$ and $t > 0$, with boundary conditions

$$\begin{cases} w(0,t) = \xi(0,t) = 0, \\ w'(1,t) + \xi(1,t) = 0, \\ \xi'(1,t) = 0 \end{cases} \tag{2.4}$$

for $t > 0$.

Now we will rewrite (2.3) in the form of operator equation (2.2), namely

$$\dot{z} = \begin{pmatrix} \dot{w} \\ \dot{\xi} \\ \ddot{w} \\ \ddot{\xi} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ -A_\gamma & 0 \end{pmatrix}}_{\mathcal{A}} \begin{pmatrix} w \\ \xi \\ \dot{w} \\ \dot{\xi} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -r-x \\ 1 \end{pmatrix} u(t), \tag{2.5}$$

where $I : D\left(A_\gamma^{\frac{1}{2}}\right) \to H = L^2\left((0,1),\mathbb{R}^2\right)$ is embedding operator, $A_\gamma : D\left(A_\gamma\right) \to H$ is linear operator defined by

$$A_\gamma \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y'' - z' \\ -\gamma^2 z'' + y' + z \end{pmatrix} \tag{2.6}$$

for $\begin{pmatrix} y \\ z \end{pmatrix} \in D\left(A_\gamma\right)$, where

$$D\left(A_\gamma\right) = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in H^2((0,1),\mathbb{R}^2) \,\middle|\, \begin{array}{l} y(0) = z(0) = 0 \\ y'(1) + z(1) = z'(1) = 0 \end{array} \right\} \tag{2.7}$$

and domain of the operator $\mathcal{A}$ is given by $D(\mathcal{A}) = D\left(A_\gamma\right) \times D\left(A_\gamma^{\frac{1}{2}}\right) \subset \mathcal{H} = D\left(A_\gamma^{\frac{1}{2}}\right) \times H$ (see [34]). Note that $A_\gamma^{\frac{1}{2}}$ is a square root of A_γ, where its existence is guaranteed by the fact that A_γ is self-adjoint and positive definite operator. $A_\gamma^{\frac{1}{2}}$ is also self-adjoint and positive definite (see Lemma 1.27).

2.3 Operator Equation of Damped Beam

One of possible extension of the model (2.5) of a rotating Timoshenko beam can be considering the effect of introducing damping to the model. We can introduce damping operator in many ways, but here we use three particular types of damping operators which were analyzed in [63–66].

Again, following [34], we will consider operator equation of the form

$$\dot{z} = \begin{pmatrix} \dot{w} \\ \dot{\xi} \\ \ddot{w} \\ \ddot{\xi} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ -A_\gamma & -B_i \end{pmatrix}}_{\mathcal{A}_i} \begin{pmatrix} w \\ \xi \\ \dot{w} \\ \dot{\xi} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -r-x \\ 1 \end{pmatrix} u(t) \tag{2.8}$$

for $i = 1, 2, 3$, where I, A_γ are defined as previously and $B_i : D(B_i) \to H$ are symmetric distributed damping operators, $D(\mathcal{A}_i) \subseteq D(\mathcal{A})$. The following damping operators were considered,

$$B_1 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \nu^2 z \end{pmatrix}, \tag{2.9}$$

where $D\left(B_1\right) = D\left(A_\gamma^{\frac{1}{2}}\right) \supset D\left(A_\gamma\right)$,

$$B_2 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -\mu^2 y'' \\ 0 \end{pmatrix}, \tag{2.10}$$

with $D\left(B_2\right) = D\left(A_\gamma\right)$. The last considered case concerns the damping operator B_3, which is an additive combination of B_1 and B_2, of the form

$$B_3 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -\mu^2 y'' \\ \nu^2 z \end{pmatrix}, \tag{2.11}$$

with $D(B_3) = D(B_2) = D(A_\gamma)$.

The main results of stability analysis in Chapter 3 are related to operator B_1, while operators B_2 and B_3 will be used to compare the obtained results.

2.4 Cantilever Beam Model

The observability problem, described in Chapter 4, will be solved on a modified model, the so-called cantilever beam model. The main difference is the elimination of the rotating disk to which the beam is clamped. As a result we obtain a rigid beam that is fixed to a support, usually presented as a vertical structure (e.g. wall) and the beam's other end is free (see Fig. 2.5 and 2.6). We additionally assume that the parameter γ describing the physical properties of the beam material is equal to 1, i.e. $\gamma = 1$. In this case, the system of partial differential equations has the following form

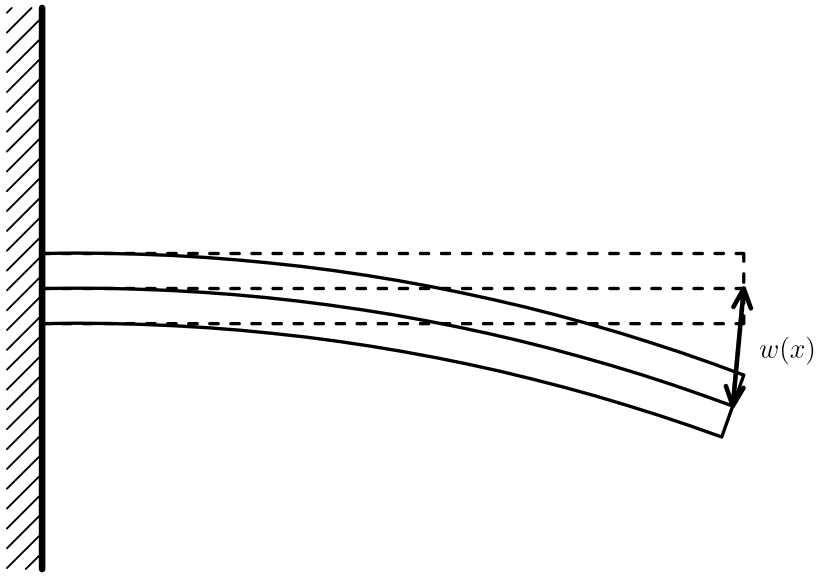

Figure 2.5: Deflection of the center line of the cantilever beam

$$\begin{cases} \ddot{w}(x,t) - w''(x,t) - \xi'(x,t) = 0, \\ \ddot{\xi}(x,t) - \xi''(x,t) + w'(x,t) + \xi(x,t) = 0 \end{cases} \tag{2.12}$$

for $x \in (0,1)$ and $t > 0$, with clamped-free boundary conditions

$$\begin{cases} w(0,t) = \xi(0,t) = 0, \\ w'(1,t) + \xi(1,t) = 0, \\ \xi'(1,t) = 0 \end{cases} \tag{2.13}$$

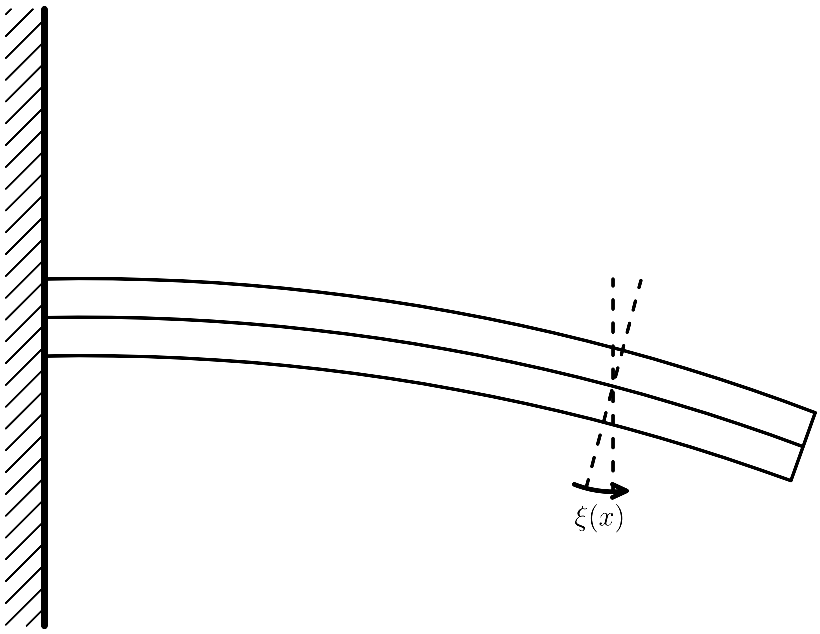

Figure 2.6: The rotation angle of the cross-section area of the cantilever beam

for $t > 0$.

Operator equation has the following form

$$\dot{z} = \begin{pmatrix} \dot{w} \\ \dot{\xi} \\ \ddot{w} \\ \ddot{\xi} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ -A_1 & 0 \end{pmatrix}}_{\mathcal{A}} \begin{pmatrix} w \\ \xi \\ \dot{w} \\ \dot{\xi} \end{pmatrix}, \tag{2.14}$$

where $I : D\left(A_1^{\frac{1}{2}}\right) \to H$ is embedding operator, $A_1 : D\left(A_1\right) \to H$ is linear operator defined by (2.6) for $\gamma = 1$, i.e.

$$A_1 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y'' - z' \\ -z'' + y' + z \end{pmatrix}, \tag{2.15}$$

where $D\left(A_1\right)$ is defined by (2.7) with $\gamma = 1$, and domain of the operator $\mathcal{A}$ is given by $D(\mathcal{A}) = D\left(A_1\right) \times D\left(A_1^{\frac{1}{2}}\right) \subset \mathcal{H} = D\left(A_1^{\frac{1}{2}}\right) \times H$.

Chapter 3

Stability Analysis

Stability is one of the most important properties of dynamical systems. It is widely used not only in mathematics but also in related fields such as physics or chemistry, and even unrelated fields such as biology or economics. In general, stability of solutions of differential equations means that small perturbations of initial conditions lead to small perturbations of solutions.

In this chapter we analyze stability of a particular model of vibrations of Timoshenko beams with a weak (distributed) damping connected to rotations of cross-sections of the beam, of deflections of the center line of the beam, and of both. In one of the cases considered, for some values of physical parameters of the beam the optimal stability margin phenomenon may be observed, which means that under some conditions there exists an optimal value of a damping coefficient, that is a coefficient that guarantees the fastest possible decay of norms of solutions of the system.

Main results of this chapter were published in [63–67].

3.1 Various Concepts of Stability

Stability can be defined in many different ways. In the literature, there are often presented definitions of stability referring to the equilibrium point (see e.g. Lyapunov stability in [25]). Here we will consider semigroup stability approach.

We start with the differential equation

$$\dot{z}(t) = Az(t),\ z(0) = z_0,\ t \geq 0, \tag{3.1}$$

where A is the infinitesimal generator of a semigroup $T(t)$ on a complex Banach space X. We use the following stability definitions related to the C_0-semigroup notation (see e.g. [10]).

Definition 3.1. The C_0-semigroup $T(t)$ on the Banach space X is *asymptotically stable* (*strongly stable*) if

$$\lim_{t\to\infty} T(t)x = 0,\ \forall x \in X.$$

In addition, if the solutions tend to zero exponentially quickly, then we can consider the *exponential stability.*

Definition 3.2. The C_0-semigroup $T(t)$ on the Banach space X is *exponentially stable* if there exist positive constants M and ω such that

$$\|T(t)\| \leq Me^{-\omega t} \text{ for } t \geq 0.$$

The constant ω is called the *decay rate*, and the supremum over all possible values of ω is the *stability margin* of $T(t)$.

Remark 3.3. Stability margin of a exponentially stable semigroup is equal to the opposite of its growth bound (see (1.2) from Theorem 1.31.e)).

In the case of finite-dimensional state space X there are easily checkable conditions for different notions of stability of system (3.1) (see [72]).

Theorem 3.4. *Let A be a generator of semigroup $T(t)$ on a finite-dimensional space X. The following conditions are equivalent:*

a) For some $M > 0$, $\omega > 0$ and all $t \geq 0$, $\|T(t)\| \leq Me^{-\omega t}$.

b) For arbitrary $z_0 \in X$, $z(t) \to 0$ exponentially as $t \to +\infty$.

c) For arbitrary $z_0 \in X$, $\int\limits_0^{+\infty} \|z(t)\|^2 dt < +\infty$.

d) For arbitrary $z_0 \in X$, $z(t) \to 0$ as $t \to +\infty$.

e) $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < 0$.

In general, if we pass from finite- to infinite-dimensional state space then the conditions from above theorem are not equivalent (see [72]).

Theorem 3.5. *Let A be a generator of semigroup $T(t)$ on an infinite–dimensional Banach space X. Then*

a) Conditions 3.4.a), 3.4.b) and 3.4.c) are all equivalent.

b) Conditions 3.4.a), 3.4.b) and 3.4.c) are essentially stronger than 3.4.d) and 3.4.e).

c) Condition 3.4.d) does not imply, in general, condition 3.4.e), even if X is a Hilbert space.

d) Condition 3.4.e) does not imply, in general, condition 3.4.d), even if X is a Hilbert space.

Theorem 3.5 shows that analyzing stability in infinite-dimensional systems is much more complicated. In particular a proper location of spectrum does not guarantee stability (see [71]), nor existence of spectrum part on the axis (c.f. [52]) does not guarantee instability. In some cases it may depend on how fast the eigenvalues approach the imaginary axis (see [51], e.g. in the case of neutral-type systems). Thus, a more detailed analysis is required. A very important result about stability in Banach spaces was obtaind by G. M. Sklyar and V. Y. Shirman in [52]. They proved that if A is a dissipative operator fulfilling the following conditions: A is bounded, the spectrum of A has at most countable intersection with the imaginary axis and A^* has no imaginary eigenvalues, then the Cauchy problem for equation (3.1) is asymptotically stable. Y. I. Lyubich and V. Q. Phóng in [40] generalized this result to unbounded operators. Also in the same year, W. Arendt and C. J. K. Batty in [2] presented independently proof of the same theorem.

Theorem 3.6 (**Arendt-Batty-Lyubich-Phóng-Sklyar-Shirman Theorem**, [2, 40, 52])**.** *Let the operator A generate a bounded strongly continuous semigroup $T(t)$, $t \geq 0$. If the intersection of the spectrum of A with the imaginary axis is at most countable and A^* has no imaginary eigenvalues, then the Cauchy problem for equation* (3.1) *is asymptotically stable.*

Remark 3.7. From Arendt-Batty-Lyubich-Phóng-Sklyar-Shirman theorem we can deduce that the operator A also has no imaginary eigenvalues, since the Cauchy problem is asymptotically stable. Therefore, if X is the reflexive Banach space, then in the theorem we can replace the absence of the imaginary eigenvalues of A instead of A^*.

In addition, Arendt-Batty-Lyubich-Phóng-Sklyar-Shirman theorem allow us to make the following corollary for C_0-groups.

Corollary 3.8. If the spectrum of the generator A of a bounded strongly continuous group does not intersect the imaginary axis, then the Cauchy problem for equation (3.1) is asymptotically stable.

3.2 Results for Undamped Beam Model

W. Krabs and G. M. Sklyar in [28, 29, 32–35] (in [28, 29] together with V. I. Korobov, in [35] together with J. Woźniak) considered different problems of controllability and stabilizability of rotating Timoshenko beams. They solved the problem of transfering the beam from a position of rest into a position of rest under a given angle within a given time [32]. This problem is solvable, if the time of rotation prescribed is large enough. In [33] the stabilizability of the model is proved and an explicit form of the stabilizing linear feedback control is given. In [28], the authors extend the results on controllability. They showed that the control realizing the rotation of a Timoshenko beam can be found in piecewise constant and gave a construction of this control. In [29] the problem of controllability from the position of rest into an arbitrary position at some given time is investigated. The problem was solved using Ullirch theorem, which is a generalization of Paley-Wiener theorem. In [35] the condition of exact controllability under the assumption that the parameter γ appearing in the model equation is rational is given.

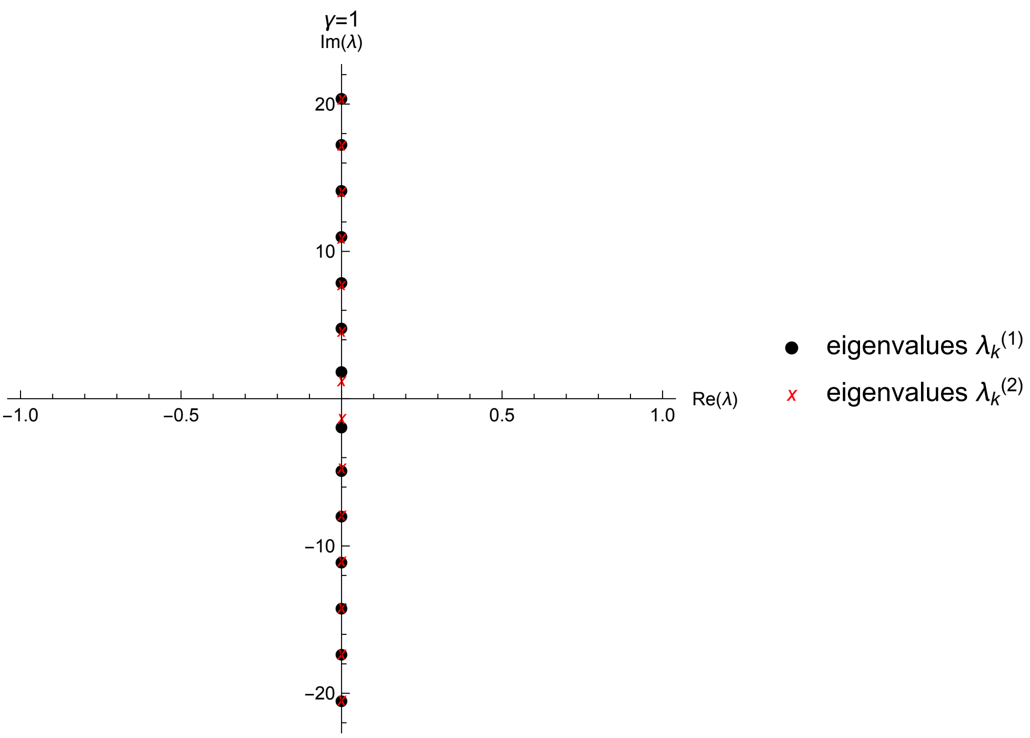

Figure 3.1: Spectrum of the operator $\mathcal{A}$ for $\gamma = 1$—two families of points approaching each other (black circles denote family $\lambda_k^{(1)}$, red crosses denote family $\lambda_k^{(2)}$)

For stability analysis, an important result was obtained in [33]. It was proved there that the undamped model of a slowly rotating Timoshenko beam is unstable. Furthermore, spectral analysis of the operator $\mathcal{A}$ defined in (2.5) allowed to formulate the following result [32, 34].

Theorem 3.9. *The eigenvalues of the operator $\mathcal{A}$ are two asymptotic families $\lambda_n^{(1)} = \pm\left(\gamma\frac{2k+1}{2}\pi + \varepsilon_{2k+1}\right)i$, if $n = 2k+1$ and $\lambda_n^{(2)} = \pm\left(\frac{2k+1}{2}\pi i + \varepsilon_{2k}\right)i$, if $n = 2k$, where $\lim\limits_{n\to\infty}\varepsilon_n = 0$. In particular case for $\gamma = 1$, the eigenvalues of the operator $\mathcal{A}$ are two asymptotic families $\lambda_n^{(1)} = \pm\left(\frac{2k+1}{2}\pi + \varepsilon_{2k+1}\right)i$, if $n = 2k+1$ and $\lambda_n^{(2)} = \pm\left(\frac{2k+1}{2}\pi i - \varepsilon_{2k}\right)i$, if $n = 2k$, where $0 < \varepsilon_{2k+1}, \varepsilon_{2k}$ and $\lim\limits_{n\to\infty}\varepsilon_n = 0$.*

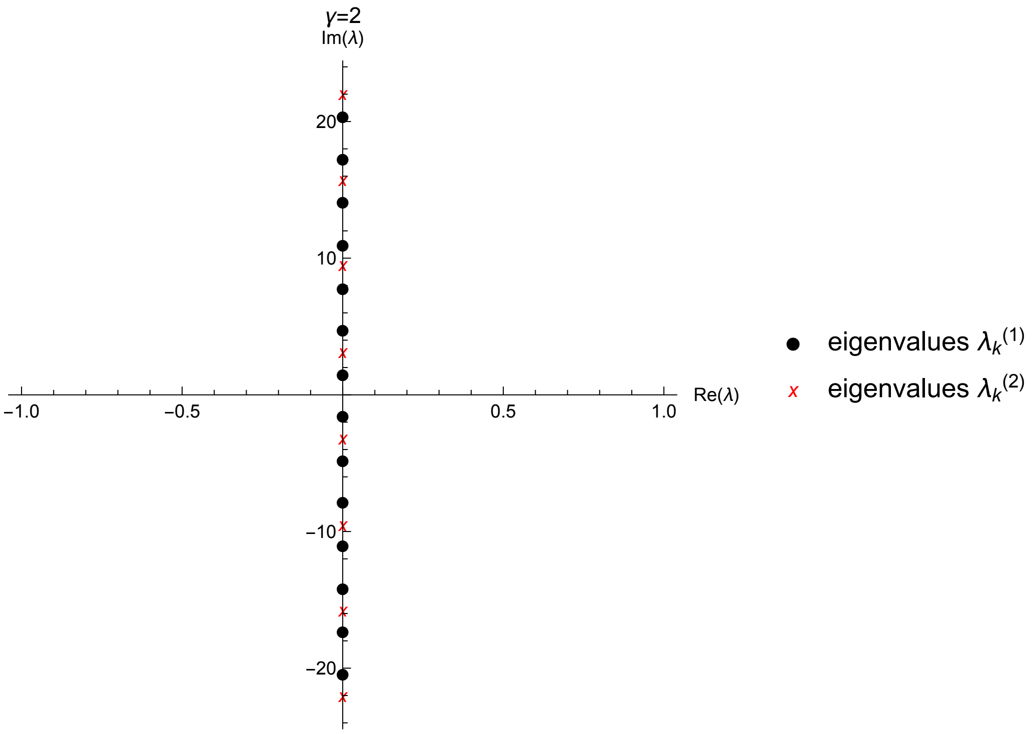

Figure 3.2: Spectrum of the operator $\mathcal{A}$ for $\gamma = 2$—two separated families of points (black circles denote family $\lambda_k^{(1)}$, red crosses denote family $\lambda_k^{(2)}$)

The behavior of the spectrum of the operator $\mathcal{A}$ is very interesting. For $\gamma = 1$ the spectrum consists of two families of points approaching each other (see Fig. 3.1), for some values of γ, e.g. $\gamma = 2$, the spectrum consists of two separated families of points (see Fig. 3.2) and for some values of γ, e.g. $\gamma = 3$, the spectrum consists of two families of points and some, but not all, subfamilies are asymptotically close (see Fig. 3.3).

In the following sections, Theorem 3.9 will be used to compare how the spectrum of the operator will change with the damping effect included.

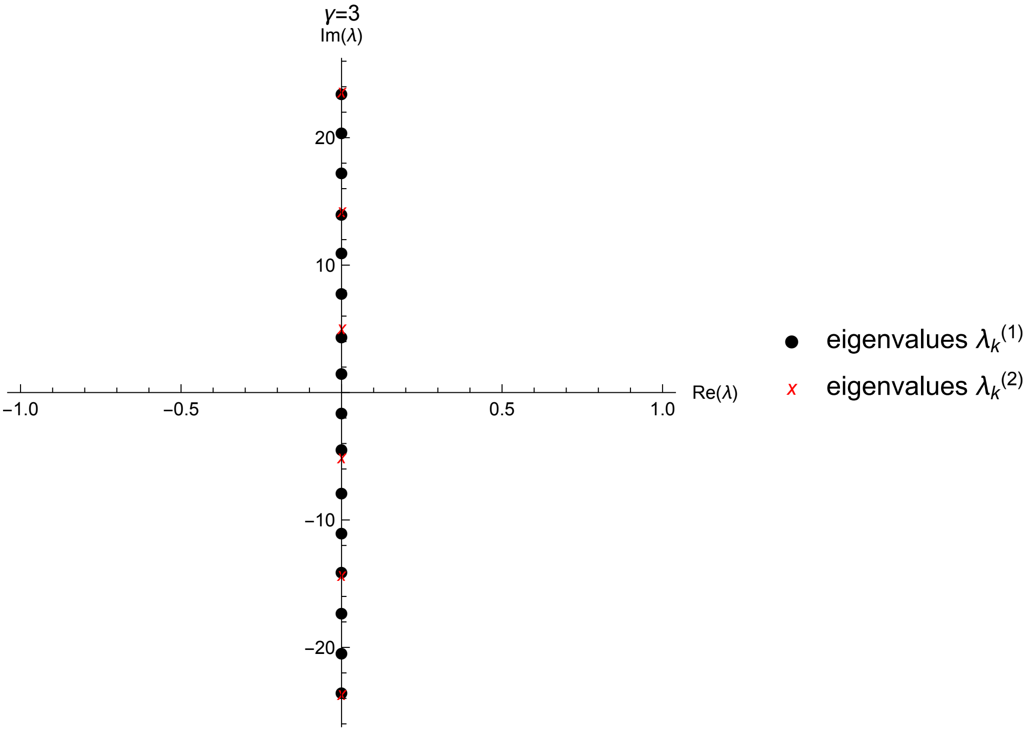

Figure 3.3: Spectrum of the operator $\mathcal{A}$ for $\gamma = 3$—two families of points, some subfamilies are asymptotically close (black circles denote family $\lambda_k^{(1)}$, red crosses denote family $\lambda_k^{(2)}$)

3.3 Spectral Properties of the Operator of Damped Beam

In this section, we derive the general spectral properties of the operator $\mathcal{A}_1$, defined by (2.8) with damping operator B_1 from (2.9); operator $\mathcal{A}_1$ is neither self-adjoint nor skew-adjoint. Namely, we prove the compactness of the resolvent. As a result we obtain only the point spectrum of the operator. Then, we observe that $\mathcal{A}_1$ generate a contraction group.

At the beginning we remind the form of operator $\mathcal{A}_1$:

$$\mathcal{A}_1 \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{pmatrix} = \begin{pmatrix} y_3(x) \\ y_4(x) \\ y_1''(x) + y_2'(x) \\ \gamma^2 y_2''(x) - y_1'(x) - y_2(x) - \nu^2 y_4(x) \end{pmatrix}, \tag{3.2}$$

with a domain

$$D(\mathcal{A}_1) = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \in H^2 \times H^1 \,\middle|\, \begin{array}{l} y_1(0) = y_2(0) = 0 \\ y_1'(1) + y_2(1) = y_2'(1) = 0 \\ y_3(0) = y_4(0) = 0 \end{array} \right\} \subset \mathcal{H},$$

where $H^2 = H^2((0,1),\mathbb{R}^2)$ and $H^1 = H^1((0,1),\mathbb{R}^2)$. Operator $\mathcal{A}_1$ is closed and densely defined. We proceed with the following lemma.

Lemma 3.10. *Operator $\mathcal{A}_1$ is invertible and the inverse operator $\mathcal{A}_1^{-1}$ is compact.*

Proof. We consider the following equation,

$$\mathcal{A}_1 \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{pmatrix} = \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ g_4(x) \end{pmatrix}, \tag{3.3}$$

where $(y_1(x), y_2(x), y_3(x), y_4(x))^T \in D(\mathcal{A}_1)$ and $(g_1(x), g_2(x), g_3(x), g_4(x))^T \in \mathcal{H}$, and T denotes transpose. Using the form (3.2) of operator $\mathcal{A}_1$, equation (3.3) reads

$$\begin{cases} y_3(x) = g_1(x), \\ y_4(x) = g_2(x), \\ y_1''(x) + y_2'(x) = g_3(x), \\ \gamma^2 y_2''(x) - y_1'(x) - y_2(x) - \nu^2 y_4(x) = g_4(x). \end{cases} \tag{3.4}$$

From the third of these equations and domain condition $y_1'(1) + y_2(1) = 0$ we obtain

$$y_1'(x) + y_2(x) = -\int_x^1 g_3(s)ds, \; x \in [0,1]. \tag{3.5}$$

The relation (3.5) and the second and the fourth equation give

$$y_2''(x) = \frac{1}{\gamma^2}\left(-\int_x^1 g_3(s)ds + \nu^2 g_2(x) + g_4(x)\right), \; x \in [0,1].$$

After integration and using domain conditions $y_2'(1) = y_2(0) = 0$ we get

$$y_2(x) = \frac{1}{\gamma^2}\left(\int_0^x \int_{s_1}^1 \int_{s_2}^1 g_3(s_3)ds_3 ds_2 ds_1 - \nu^2 \int_0^x \int_{s_1}^1 g_2(s_2)ds_2 ds_1 \right.$$
$$\left. - \int_0^x \int_{s_1}^1 g_4(s_2)ds_2 ds_1\right), \; x \in [0,1].$$

Now substituting $y_2(x)$ into (3.5) and taking account of domain condition $y_1(0)=0$ we obtain

$$y_1(x) = -\int_0^x \int_{s_1}^1 g_3(s_2)ds_2ds_1 - \frac{1}{\gamma^2}\left(\int_0^x \int_0^{s_1}\int_{s_2}^1\int_{s_3}^1 g_3(s_4)ds_4ds_3ds_2ds_1 \right.$$
$$\left. -\nu^2\int_0^x \int_0^{s_1}\int_{s_2}^1 g_2(s_3)ds_3ds_2ds_1 - \int_0^x \int_0^{s_1}\int_{s_2}^1 g_4(s_3)ds_3ds_2ds_1\right).$$

Since $(g_3(x),g_4(x))^T \in H = L^2\left((0,1),\mathbb{R}^2\right)$ and $y_1(x)$ and $y_2(x)$ are integrals of $g_2(x), g_3(x)$ and $g_4(x)$, so Sobolev's Embedding Theorem (see e.g. [14]) asserts that $\mathcal{A}^{-1}$ is a compact operator on $\mathcal{H}$.

□

Corollary 3.11. Operator $\mathcal{A}_1$ has a compact resolvent and the spectrum $\sigma\left(\mathcal{A}_1\right)$ is point-wise.

Following the authors of [34] let us introduce scalar product in considered Hilbert space $\mathcal{H}$

$$\langle z_1, z_2\rangle_{\mathcal{H}} = \left\langle A^{\frac{1}{2}}v_1, A^{\frac{1}{2}}v_2\right\rangle_H + \langle w_1, w_2\rangle_H \tag{3.6}$$

for all $z_1 = (v_1,w_1)^T$ and $z_2=(v_2,w_2)^T$ in $\mathcal{H}$ with accompanying norm $\|z\|_{\mathcal{H}}^2 = \langle z,z\rangle_{\mathcal{H}}$. Further it follows, for every $z=(y_1,y_2,y_3,y_4)^T\in\mathcal{H}$,

$$\begin{aligned}
\langle \mathcal{A}z, z\rangle_{\mathcal{H}} &= \left\langle A^{\frac{1}{2}}\begin{pmatrix} y_3 \\ y_4\end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}\right\rangle_H - \left\langle A\begin{pmatrix} y_1 \\ y_2\end{pmatrix}, \begin{pmatrix} y_3 \\ y_4\end{pmatrix}\right\rangle_H \\
&\quad - \left\langle B\begin{pmatrix} y_3 \\ y_4\end{pmatrix}, \begin{pmatrix} y_3 \\ y_4\end{pmatrix}\right\rangle_H \\
&= \left\langle A^{\frac{1}{2}}\begin{pmatrix} y_3 \\ y_4\end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}\right\rangle_H - \left\langle A^{\frac{1}{2}}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} y_3 \\ y_4\end{pmatrix}\right\rangle_H \\
&\quad - \nu^2\int_0^1 y_4^2(x)dx \\
&= \overline{\left\langle A^{\frac{1}{2}}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} y_3 \\ y_4\end{pmatrix}\right\rangle_H} - \left\langle A^{\frac{1}{2}}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} y_3 \\ y_4\end{pmatrix}\right\rangle_H \\
&\quad - \nu^2\int_0^1 y_4^2(x)dx.
\end{aligned}$$

Observe that

$$\mathrm{Re}\left(\overline{\left\langle A^{\frac{1}{2}}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} y_3 \\ y_4\end{pmatrix}\right\rangle_H} - \left\langle A^{\frac{1}{2}}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} y_3 \\ y_4\end{pmatrix}\right\rangle_H\right) = 0$$

thus

$$\operatorname{Re}\langle \mathcal{A}z, z\rangle_{\mathcal{H}} = -\nu^2 \int_0^1 y_4^2(x)dx \leq 0. \tag{3.7}$$

We perform similar calculations for operator $-\mathcal{A} - \nu^2 I$

$$\begin{aligned}
\left\langle \left(-\mathcal{A} - \nu^2 I\right) z, z\right\rangle_{\mathcal{H}} = &-\left\langle A^{\frac{1}{2}} \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}, A^{\frac{1}{2}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\rangle_H \\
&+ \left\langle A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}\right\rangle_H \\
&+ \left\langle B \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}, \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}\right\rangle_H \\
&- \nu^2 \left\langle A^{\frac{1}{2}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, A^{\frac{1}{2}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\rangle_H \\
&- \nu^2 \left\langle \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}, \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}\right\rangle_H .
\end{aligned}$$

Again, we observe that

$$\operatorname{Re}\left(-\left\langle A^{\frac{1}{2}} \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}, A^{\frac{1}{2}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\rangle_H + \left\langle A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}\right\rangle_H\right) = 0,$$

so

$$\begin{aligned}
\operatorname{Re}\left\langle \left(-\mathcal{A} - \nu^2 I\right) z, z\right\rangle_{\mathcal{H}} &= \nu^2 \int_0^1 y_4^2(x)dx - \nu^2 \left\| A^{\frac{1}{2}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\|_H^2 \\
&\quad - \nu^2 \int_0^1 y_3^2(x)dx - \nu^2 \int_0^1 y_4^2(x)dx \\
&= -\nu^2 \left\| A^{\frac{1}{2}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\|_H^2 - \nu^2 \int_0^1 y_3^2(x)dx \leq 0.
\end{aligned}$$

This gives us the proof of the following lemma.

Lemma 3.12. *Operators $\mathcal{A}_1$ and $-\mathcal{A}_1 - \nu^2 I$ are dissipative.*

Lemma 3.10, Corollary 3.11 and Lemma 3.12 allow us to formulate the following theorem.

Theorem 3.13. *Operators $\mathcal{A}_1$ and $-\mathcal{A}_1 - \nu^2 I$ generate contraction semigroups. Hence, $\mathcal{A}_1$ is the infinitesimal generator of a C_0-group.*

Proof. The proof of the first part of the theorem for operators $\mathcal{A}_1$ and $-\mathcal{A}_1 - \nu^2 I$ is a direct consequence of Lumer-Phillips theorem (see Theorem 1.49). Furthermore, using Phillips theorem (see Theorem 1.50) we know that $(-\mathcal{A}_1 - \nu^2 I) + \nu^2 I = -\mathcal{A}_1$ generates a semigroup. Thus, the operators $\mathcal{A}_1$ and $-\mathcal{A}_1$ generate C_0-semigroups which means from Lemma 1.53 that the operator $\mathcal{A}_1$ generate C_0-group, what completes the proof of the theorem. $\square$

Existence of solution and well-posedness of system (2.8) for $i = 1$ follows from Theorem 3.13.

3.4 Asymptotic Stability of the System

Now we are able to prove the following theorem about asymptotic stability of the considered system.

Theorem 3.14. *C_0-semigroup $T(t)$ generated by $\mathcal{A}_1$ is asymptotically stable.*

Proof. Using Theorem 3.6 and Remark 3.7, it is sufficient to show that $\operatorname{Re}\lambda < 0$ for any $\lambda \in \sigma(\mathcal{A}_1)$. The dissipativity of $\mathcal{A}_1$ (Lemma 3.12) implies $\operatorname{Re}\lambda \leq 0$ for any $\lambda \in \sigma(\mathcal{A}_1)$. Hence we need to show that there are no eigenvalues on the imaginary axis. According to Lemma 3.10, we have $0 \notin \sigma(\mathcal{A}_1)$.

We consider the following eigenvalue problem

$$\mathcal{A}_1 \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{pmatrix} = \lambda \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{pmatrix}, \tag{3.8}$$

where $(y_1(x), y_2(x), y_3(x), y_4(x))^T \in D(\mathcal{A}_1)$ is an eigenvector corresponding to $\lambda = i\mu$, $\mu \in \mathbb{R}$, $\mu \neq 0$. Using the form (3.2) of operator $\mathcal{A}_1$, equation (3.8) reads

$$\begin{cases} y_3(x) = \lambda y_1(x), \\ y_4(x) = \lambda y_2(x), \\ y_1''(x) + y_2'(x) = \lambda y_3(x), \\ \gamma^2 y_2''(x) - y_1'(x) - y_2(x) - \nu^2 y_4(x) = \lambda y_4(x), \end{cases} \tag{3.9}$$

and using (3.7) we obtain

$$\begin{aligned} 0 = \operatorname{Re}\lambda \left\|(y_1, y_2, y_3, y_4)^T\right\|_{\mathcal{H}}^2 &= \operatorname{Re}\langle \mathcal{A}_1 (y_1, y_2, y_3, y_4)^T, (y_1, y_2, y_3, y_4)^T \rangle_{\mathcal{H}} \\ &= -\nu^2 \int_0^1 y_4^2(x)dx \leq 0. \end{aligned}$$

Since $\nu^2 > 0$, it must be $y_4(x) \equiv 0$. Then from the second equation of (3.9) we obtain $y_2(x) \equiv 0$. Now our eigenvalue problem (3.9) reads

$$\begin{cases} \quad y_3(x) = \lambda y_1(x), \\ \quad y_1''(x) = \lambda^2 y_1(x), \\ -y_1'(x) = 0. \end{cases}$$

From the last of these equations we deduce that $y_1(x) \equiv$ const and the form of $D(\mathcal{A}_1)$ (i.a. $y_1(0) = 0$) implies that $y_1(x) \equiv 0$ and $y_3(x) \equiv 0$. We have shown that $(y_1(x), y_2(x), y_3(x), y_4(x))^T = 0$ which implies that there are no eigenvalues on the imaginary axis, what finishes the proof.

□

3.5 General Form of a Spectral Equation

This section is devoted to find a general form of a spectral equation of operator $\mathcal{A}_1$. We consider two main cases of physical parameters of the beam, $\gamma^2 > 1$ and $\gamma^2 = 1$, and different values of damping coefficient ν.

At the beginning we prove the following lemma, which helps us in further considerations.

Lemma 3.15. *Let $\gamma^2 \geq 1$. Spectral equation $\mathcal{P}(\lambda) = 0$ of system (2.8) for $i = 1$ can be written in the form*

$$\mathcal{P}(\lambda) = (a_{23}(1,\lambda) + a_{33}(1,\lambda))a_{44}(1,\lambda) - (a_{24}(1,\lambda) + a_{34}(1,\lambda))a_{43}(1,\lambda) = 0,$$

where $a_{ij}(1,\lambda)$ are elements of matrix exponential of

$$M_1(\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda^2 & 0 & 0 & -1 \\ 0 & \frac{\lambda^2+\nu^2\lambda+1}{\gamma^2} & \frac{1}{\gamma^2} & 0 \end{pmatrix}.$$

Proof. We have the following eigenvalue problem

$$\mathcal{A}_1 \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{pmatrix} = \lambda \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{pmatrix} \tag{3.10}$$

and

$$\begin{cases} y_1(0) = y_2(0) = 0, \\ y_1'(1) + y_2(1) = 0, \\ \qquad\quad y_2'(1) = 0 \end{cases}$$

for $x \in (0,1)$. Using the form of operator $\mathcal{A}_1$, eigenvalue problem (3.10) reads

$$\begin{cases} y_3(x) = \lambda y_1(x), \\ y_4(x) = \lambda y_2(x), \\ y_1''(x) + y_2'(x) = \lambda y_3(x), \\ \gamma^2 y_2''(x) - y_1'(x) - y_2(x) - \nu^2 y_4(x) = \lambda y_4(x). \end{cases}$$

Thus, we obtain

$$\begin{cases} y_1''(x) = \lambda^2 y_1(x) - y_2'(x), \\ y_2''(x) = \dfrac{\lambda^2 + \nu^2\lambda + 1}{\gamma^2} y_2(x) + \dfrac{1}{\gamma^2} y_1'(x). \end{cases} \tag{3.11}$$

In order to solve system (3.11) we introduce a standard change of variables,

$$\begin{cases} z_1 = y_1, \\ z_2 = y_2, \\ z_3 = y_1', \\ z_4 = y_2', \end{cases}$$

and put system (3.11) into first-order form,

$$\frac{d}{dx}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda^2 & 0 & 0 & -1 \\ 0 & \frac{\lambda^2+\nu^2\lambda+1}{\gamma^2} & \frac{1}{\gamma^2} & 0 \end{pmatrix}}_{M_1(\lambda)} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \tag{3.12}$$

with conditions

$$\begin{cases} z_1(0) = z_2(0) = 0, \\ z_2(1) + z_3(1) = 0, \\ z_4(1) = 0. \end{cases} \tag{3.13}$$

The matrix exponential of $M_1(\lambda)$ with respect to x we denote as

$$e^{M_1(\lambda)x} = \begin{pmatrix} a_{11}(x,\lambda) & a_{12}(x,\lambda) & a_{13}(x,\lambda) & a_{14}(x,\lambda) \\ a_{21}(x,\lambda) & a_{22}(x,\lambda) & a_{23}(x,\lambda) & a_{24}(x,\lambda) \\ a_{31}(x,\lambda) & a_{32}(x,\lambda) & a_{33}(x,\lambda) & a_{34}(x,\lambda) \\ a_{41}(x,\lambda) & a_{42}(x,\lambda) & a_{43}(x,\lambda) & a_{44}(x,\lambda) \end{pmatrix}.$$

General solution of system (3.12)–(3.13), for initial conditions $z_1(0,\lambda) = 0$, $z_2(0,\lambda) = 0$, $z_3(0,\lambda) = \gamma(\lambda)$ and $z_4(0,\lambda) = \delta(\lambda)$, is given by

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = e^{M_1(\lambda)x} \begin{pmatrix} 0 \\ 0 \\ \gamma(\lambda) \\ \delta(\lambda) \end{pmatrix}, \tag{3.14}$$

where $\gamma(\lambda)$, $\delta(\lambda)$ are unknown functions.

The boundary conditions (3.13) lead to the conditions

$$C\begin{pmatrix} 0 \\ 0 \\ \gamma(\lambda) \\ \delta(\lambda) \end{pmatrix} = 0,$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_{31}(1,\lambda) & b_{32}(1,\lambda) & b_{33}(1,\lambda) & b_{34}(1,\lambda) \\ a_{41}(1,\lambda) & a_{42}(1,\lambda) & a_{43}(1,\lambda) & a_{44}(1,\lambda) \end{pmatrix}$$

and

$$\begin{aligned} b_{31}(1,\lambda) &= a_{21}(1,\lambda) + a_{31}(1,\lambda), \\ b_{32}(1,\lambda) &= a_{22}(1,\lambda) + a_{32}(1,\lambda), \\ b_{33}(1,\lambda) &= a_{23}(1,\lambda) + a_{33}(1,\lambda), \\ b_{34}(1,\lambda) &= a_{24}(1,\lambda) + a_{34}(1,\lambda). \end{aligned}$$

A necessary and sufficient condition for this system to have a nontrivial solution is that

$$\det C = 0,$$

which is equivalent to solving

$$(a_{23}(1,\lambda) + a_{33}(1,\lambda))\, a_{44}(1,\lambda) - (a_{24}(1,\lambda) + a_{34}(1,\lambda))\, a_{43}(1,\lambda) = 0,$$

which finishes the proof of Lemma 3.15. $\square$

Remark 3.16. Lemma 3.15 is also true for operators $\mathcal{A}_2$ and $\mathcal{A}_3$, but instead of matrix exponential of $M_1(\lambda)$ we have to use matrix exponential of $M_2(\lambda)$, where

$$M_2(\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\lambda^2}{1+\mu^2\lambda} & 0 & 0 & -\frac{1}{1+\mu^2\lambda} \\ 0 & \frac{\lambda^2+1}{\gamma^2} & \frac{1}{\gamma^2} & 0 \end{pmatrix}$$

or $M_3(\lambda)$, where

$$M_3(\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\lambda^2}{1+\mu^2\lambda} & 0 & 0 & -\frac{1}{1+\mu^2\lambda} \\ 0 & \frac{\lambda^2+\nu^2\lambda+1}{\gamma^2} & \frac{1}{\gamma^2} & 0 \end{pmatrix}$$

respectively.

3.6 Approximations of a Spectral Equation

Now we find asymptotic formulas for eigenvalues of the operator $\mathcal{A}_1$. We use the asymptotic behavior method described in [34]. The main idea of this method is to find the roots of an asymptotically approximate spectral equation. Next, we prove lemma about approximation of roots, i.e., we show that in a neighborhood of each sufficiently large root of approximate spectral equation there is at least one root of original spectral equation. In particular, one can easily see that $\mathcal{A}_1$ has only a point spectrum of finite multiplicity. Furthermore, we show that eigenvectors of operator $\mathcal{A}_1$ forms a complete set in $\mathcal{H}$. Then, using Zwart Theorem (see [75]) we observe that a semigroup $\mathcal{T}(t)$ generated by $\mathcal{A}_1$ satisfies the spectrum determined growth condition.

At the beginning we consider the case of $\gamma^2 > 1$.

Theorem 3.17. *Let $\gamma^2 > 1$. For any value of a damping constant $0 < \nu < \infty$ the eigenvalues of the operator $\mathcal{A}_1$ are two asymptotic families $\lambda_k^{(1)} = -\frac{1}{2}\nu^2 + \gamma\frac{2k+1}{2}\pi i + \varepsilon_k^{(1)}$ and $\lambda_k^{(2)} = \frac{2k+1}{2}\pi i + \varepsilon_k^{(2)}$, where $\lim\limits_{k\to\infty} \varepsilon_k^{(i)} = 0$.*

Proof. We use Lemma 3.15 to determine spectral equation. To use it we need the following

$$
\begin{aligned}
a_{23}(1,\lambda) &= \frac{1}{\lambda\sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}} \left(-\cosh(\sigma_1(\lambda)) + \cosh(\sigma_2(\lambda))\right), \\
a_{24}(1,\lambda) &= \frac{\gamma\left(-\lambda + \lambda\gamma^2 - \nu^2 + \sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}\right)}{\sqrt{2}\sqrt{\lambda}\sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}} \\
&\quad \cdot \frac{1}{\sqrt{\lambda + \lambda\gamma^2 + \nu^2 - \sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}}} \sinh\left(\sigma_1(\lambda)\right) \\
&\quad + \frac{\gamma\left(\lambda - \lambda\gamma^2 + \nu^2 + \sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}\right)}{\sqrt{2}\sqrt{\lambda}\sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}} \\
&\quad \cdot \frac{1}{\sqrt{\lambda + \lambda\gamma^2 + \nu^2 + \sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}}} \sinh\left(\sigma_2(\lambda)\right), \\
a_{33}(1,\lambda) &= \frac{2 + \lambda^2 - \lambda^2\gamma^2 + \lambda\nu^2 + \lambda\sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}}{2\lambda\sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}} \cosh\left(\sigma_1(\lambda)\right) \\
&\quad + \frac{-2 - \lambda^2 + \lambda^2\gamma^2 - \lambda\nu^2 + \lambda\sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}}{2\lambda\sqrt{-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2}}
\end{aligned}
$$

$$\cdot \cosh\left(\sigma_2(\lambda)\right),$$

$$a_{34}(1,\lambda) = \frac{\gamma\sqrt{\lambda+\lambda\gamma^2+\nu^2-\sqrt{-4\gamma^2+\left(\lambda-\lambda\gamma^2+\nu^2\right)^2}}}{\sqrt{2}\sqrt{\lambda}\sqrt{-4\gamma^2+(\lambda-\lambda\gamma^2+\nu^2)^2}}\sinh\left(\sigma_1(\lambda)\right)$$

$$-\frac{\gamma\sqrt{\lambda+\lambda\gamma^2+\nu^2+\sqrt{-4\gamma^2+\left(\lambda-\lambda\gamma^2+\nu^2\right)^2}}}{\sqrt{2}\sqrt{\lambda}\sqrt{-4\gamma^2+(\lambda-\lambda\gamma^2+\nu^2)^2}}\sinh\left(\sigma_2(\lambda)\right),$$

$$a_{43}(1,\lambda) = -\frac{\sqrt{\lambda+\lambda\gamma^2+\nu^2-\sqrt{-4\gamma^2+\left(\lambda-\lambda\gamma^2+\nu^2\right)^2}}}{\sqrt{2}\sqrt{\lambda}\gamma\sqrt{-4\gamma^2+(\lambda-\lambda\gamma^2+\nu^2)^2}}\sinh\left(\sigma_1(\lambda)\right)$$

$$+\frac{\sqrt{\lambda+\lambda\gamma^2+\nu^2+\sqrt{-4\gamma^2+\left(\lambda-\lambda\gamma^2+\nu^2\right)^2}}}{\sqrt{2}\sqrt{\lambda}\gamma\sqrt{-4\gamma^2+(\lambda-\lambda\gamma^2+\nu^2)^2}}\sinh\left(\sigma_2(\lambda)\right)$$

$$a_{44}(1,\lambda) = \frac{-\lambda+\lambda\gamma^2-\nu^2+\sqrt{-4\gamma^2+(\lambda-\lambda\gamma^2+\nu^2)^2}}{2\sqrt{-4\gamma^2+(\lambda-\lambda\gamma^2+\nu^2)^2}}\cosh\left(\sigma_1(\lambda)\right)$$

$$+\frac{\lambda-\lambda\gamma^2+\nu^2+\sqrt{-4\gamma^2+(\lambda-\lambda\gamma^2+\nu^2)^2}}{2\sqrt{-4\gamma^2+(\lambda-\lambda\gamma^2+\nu^2)^2}}\cosh\left(\sigma_2(\lambda)\right),$$

where

$$\begin{cases} \sigma_1(\lambda) = \dfrac{\sqrt{\lambda}\sqrt{\lambda+\lambda\gamma^2+\nu^2-\sqrt{-4\gamma^2+\left(\lambda-\lambda\gamma^2+\nu^2\right)^2}}}{\sqrt{2}\gamma}, \\ \sigma_2(\lambda) = \dfrac{\sqrt{\lambda}\sqrt{\lambda+\lambda\gamma^2+\nu^2+\sqrt{-4\gamma^2+\left(\lambda-\lambda\gamma^2+\nu^2\right)^2}}}{\sqrt{2}\gamma}. \end{cases}$$

Now we can consider spectral equation $\mathcal{P}(\lambda) = 0$, with

$$\begin{aligned} \mathcal{P}(\lambda) = \sigma_3(\lambda) + \sigma_4(\lambda)\cosh(\sigma_1(\lambda))\cosh(\sigma_2(\lambda)) \\ + \sigma_5(\lambda)\sinh(\sigma_1(\lambda))\sinh(\sigma_2(\lambda)), \end{aligned} \tag{3.15}$$

where

$$\begin{cases} \sigma_3(\lambda) = \dfrac{2\gamma^2}{(2\gamma - \lambda + \lambda\gamma^2 - \nu^2)(2\gamma + \lambda - \lambda\gamma^2 + \nu^2)}, \\ \sigma_4(\lambda) = \dfrac{2\gamma^2 - (\lambda - \lambda\gamma^2 + \nu^2)^2}{(2\gamma - \lambda + \lambda\gamma^2 - \nu^2)(2\gamma + \lambda - \lambda\gamma^2 + \nu^2)}, \\ \sigma_5(\lambda) = \dfrac{2\gamma^2(\lambda + \lambda\gamma^2 + \nu^2)}{(2\gamma - \lambda + \lambda\gamma^2 - \nu^2)(2\gamma + \lambda - \lambda\gamma^2 + \nu^2)} \\ \qquad\quad \cdot \dfrac{1}{\sqrt{(\lambda + \lambda\gamma^2 + \nu^2)^2 - \left(-4\gamma^2 + (\lambda - \lambda\gamma^2 + \nu^2)^2\right)}}. \end{cases}$$

To determine location of eigenvalues, it is sufficient to solve spectral equation $\mathcal{P}(\lambda) = 0$. As one can see, solving this equation is nontrivial, we will use asymptotic behavior method from [34]. Then, for sufficiently large $|\lambda|$, we obtain

$$\begin{cases} \sigma_1(\lambda) = \dfrac{1}{\gamma}\lambda + \dfrac{1}{2\gamma}\nu^2 + \left(\dfrac{\gamma}{2(-1+\gamma^2)} - \dfrac{\nu^4}{2\gamma}\right)\dfrac{1}{\lambda} + \varphi_1\left(\dfrac{1}{\lambda^2}\right), \\ \sigma_2(\lambda) = \lambda - \dfrac{1}{2(-1+\gamma^2)}\dfrac{1}{\lambda} + \varphi_2\left(\dfrac{1}{\lambda^2}\right), \\ \sigma_3(\lambda) = -\dfrac{2\gamma^2}{(-1+\gamma^2)^2}\dfrac{1}{\lambda^2} - \dfrac{4\gamma^2\nu^2}{(-1+\gamma^2)^3}\dfrac{1}{\lambda^3} + \varphi_3\left(\dfrac{1}{\lambda^4}\right), \\ \sigma_4(\lambda) = 1 + \dfrac{2\gamma^2}{(-1+\gamma^2)^2}\dfrac{1}{\lambda^2} + \dfrac{4\gamma^2\nu^2}{(-1+\gamma^2)^3}\dfrac{1}{\lambda^3} + \varphi_4\left(\dfrac{1}{\lambda^4}\right), \\ \sigma_5(\lambda) = -\dfrac{\gamma(1+\gamma^2)}{(-1+\gamma^2)^2}\dfrac{1}{\lambda^2} + \dfrac{\gamma(-3-6\gamma^2+\gamma^4)\nu^2}{2(-1+\gamma^2)^3}\dfrac{1}{\lambda^3} + \varphi_5\left(\dfrac{1}{\lambda^4}\right), \end{cases}$$

where each $\varphi_i(\cdot)$ is an analytic function in a neighborhood of 0 with

$$\lim_{|\lambda|\to\infty} \varphi_i\left(\frac{1}{\lambda}\right) = 0.$$

Hence we can define asymptotic approximations of σ_i's as $|\lambda| \to \infty$ in the form

$$\begin{cases} \widetilde{\sigma}_1(\lambda) = \dfrac{1}{\gamma}\lambda + \dfrac{1}{2\gamma}\nu^2, \\ \widetilde{\sigma}_2(\lambda) = \lambda, \\ \widetilde{\sigma}_3(\lambda) = 0, \\ \widetilde{\sigma}_4(\lambda) = 1, \\ \widetilde{\sigma}_5(\lambda) = -\dfrac{\gamma(1+\gamma^2)}{(1+\gamma^2)^2}\dfrac{1}{\lambda^2}. \end{cases}$$

Now we consider an approximate equation of the form

$$\cosh\left(\frac{1}{\gamma}\lambda+\frac{1}{2\gamma}\nu^2\right)\cosh(\lambda)-\frac{\gamma(1+\gamma^2)}{(1+\gamma^2)^2}\frac{1}{\lambda^2}\sinh\left(\frac{1}{\gamma}\lambda+\frac{1}{2\gamma}\nu^2\right)\sinh(\lambda)=0.$$

After small calculations we can rewrite it as another approximate equation (see Lemma 3.18)

$$\cosh\left(\frac{1}{\gamma}\lambda+\frac{1}{2\gamma}\nu^2\right)\cosh(\lambda)=0. \tag{3.16}$$

We observe that (3.16) is true, when

$$\cosh\left(\frac{1}{\gamma}\lambda+\frac{1}{2\gamma}\nu^2\right)=0 \tag{3.17}$$

or

$$\cosh(\lambda)=0. \tag{3.18}$$

Let $\frac{1}{\gamma}\lambda+\frac{1}{2\gamma}\nu^2=i\tau$, then (3.17) is transferred into

$$\cos(\tau)=0,$$

which has a sequence (τ_k) of solutions of the form

$$\tau_k=\frac{2k+1}{2}\pi.$$

Thus

$$\tilde{\lambda}_k^{(1)}=-\frac{1}{2}\nu^2+\gamma\frac{2k+1}{2}\pi i.$$

Now we do a similar change of variable ($\lambda=i\tau$) and solve equation (3.18)

$$\cosh(i\tau)=0,$$

so

$$\tilde{\lambda}_k^{(2)}=\frac{2k+1}{2}\pi i.$$

Hence, solution of (3.16) consist of two families, $\tilde{\lambda}_k^{(1)}$ and $\tilde{\lambda}_k^{(2)}$, where

$$\tilde{\lambda}_k^{(1)}=-\frac{1}{2}\nu^2+\gamma\frac{2k+1}{2}\pi i,\ \tilde{\lambda}_k^{(2)}=\frac{2k+1}{2}\pi i \text{ as } k\to\infty.$$

(see Fig. 3.4). □

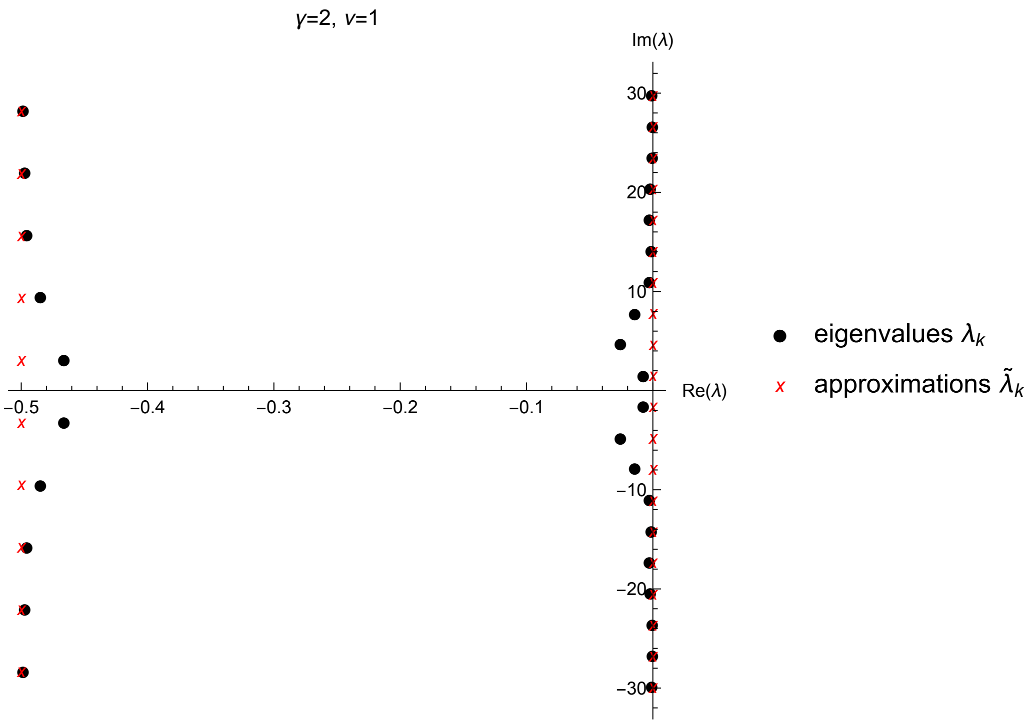

Figure 3.4: Eigenvalues in the case $\gamma = 2$ and $\nu = 1$ (black circles denote eigenvalues λ_k, red crosses denote approximations of eigenvalues $\tilde{\lambda}_k$)

The following lemma shows that in a neighborhood of each sufficiently large root of (3.16) there exist at least one root of (3.15) (compare Theorem 3.1, p. 81 in [34]).

Lemma 3.18. *For every $\delta > 0$ there exist $M > 0$ such that if $\tilde{\lambda} > M$ is a root of* (3.16)*, then there exist a root λ_0 of* (3.15) *with $\left|\tilde{\lambda} - \lambda_0\right| < \delta$.*

Proof. At the beginning we rewrite (3.15) in the form

$$\cosh(\sigma_1(\lambda))\cosh(\sigma_2(\lambda)) = \varepsilon(\lambda), \tag{3.19}$$

where

$$\varepsilon(\lambda) = \frac{1}{\sigma_4(\lambda)}\left(-\sigma_3(\lambda) - \sigma_5(\lambda)\sinh(\sigma_1(\lambda))\sinh(\sigma_2(\lambda))\right)$$

with

$$\lim_{|\lambda|\to\infty} \varepsilon(\lambda) = 0.$$

Let $\tilde{\lambda} > M$ be a root of (3.16) which is unique in the neighborhood $|\lambda - \tilde{\lambda}| < \delta$. Then there exists $\Delta > 0$ (not depending on M) such that the values of

$|\cosh(\tilde{\sigma}_2(\lambda))\cosh(\tilde{\sigma}_2\lambda)|$ cover the whole interval $[0,\Delta]$ as $\left|\lambda-\tilde{\lambda}\right|<\delta$. If in addition M is large enough, this implies that (3.19) has a root $\lambda_0 \in \left\{\lambda : \left|\lambda-\tilde{\lambda}\right|<\delta\right\}$.

□

Corollary 3.19. Eigensystem of operator $\mathcal{A}_1$ forms a complete set in $\mathcal{H}$.

Proof. From (3.14) we can infer that the eigenvectors corresponding to the eigenvalue λ can be approximated as

$$\begin{pmatrix} \tilde{y}_1(x) \\ \tilde{y}_2(x) \\ \tilde{y}_3(x) \\ \tilde{y}_4(x) \end{pmatrix} = \begin{pmatrix} -\frac{\gamma}{(\gamma^2-1)^2\lambda^3}\sinh\left(\frac{\lambda}{\gamma}x\right)+\frac{1}{\lambda}\sinh(\lambda x) \\ \frac{1}{(\gamma^2-1)^2}\left[-\cosh\left(\frac{\lambda}{\gamma}x\right)+\cosh(\lambda x)\right] \\ -\frac{\gamma}{(\gamma^2-1)^2\lambda^2}\sinh\left(\frac{\lambda}{\gamma}x\right)+\sinh(\lambda x) \\ \frac{\lambda}{(\gamma^2-1)^2}\left[-\cosh\left(\frac{\lambda}{\gamma}x\right)+\cosh(\lambda x)\right] \end{pmatrix}\gamma(\lambda)$$
$$+\begin{pmatrix} \frac{\gamma^2}{(\gamma^2-1)\lambda^2}\left[\cosh\left(\frac{\lambda}{\gamma}x\right)-\cosh(\lambda x)\right] \\ \frac{\gamma}{\lambda}\sinh\left(\frac{\lambda}{\gamma}x\right)-\frac{\gamma^2}{(\gamma^2-1)^2\lambda^3}\sinh(\lambda x) \\ \frac{\gamma^2}{(\gamma^2-1)\lambda}\left[\cosh\left(\frac{\lambda}{\gamma}x\right)-\cosh(\lambda x)\right] \\ \gamma\sinh\left(\frac{\lambda}{\gamma}x\right)-\frac{\gamma^2}{(\gamma^2-1)^2\lambda^2}\sinh(\lambda x) \end{pmatrix}\delta(\lambda)$$

which together with Lemma 3.18 implies the thesis (cf. [3]).

□

Now we proceed with the particular case of $\gamma = 1$.

Theorem 3.20. *Let $\gamma = 1$. For any value of a damping constant $0<\nu<\infty$ the operator $\mathcal{A}_1$ has the eigenvalues of the following form*

$$\begin{cases} \lambda_k^{(1)} = -\dfrac{1}{4}\nu^2+\dfrac{1}{2}\ln(y-\sqrt{y^2-1})+\dfrac{2k+1}{2}\pi i+\varepsilon_k^{(1)}, \\ \lambda_k^{(2)} = -\dfrac{1}{4}\nu^2+\dfrac{1}{2}\ln(y+\sqrt{y^2-1})+\dfrac{2k+1}{2}\pi i+\varepsilon_k^{(2)}, \end{cases}$$

where $\lim_{k\to\infty}\varepsilon_k^{(i)}=0$, and $y=y(\nu)$ is given by

$$y=\frac{\nu^4\cosh\left(\frac{\sqrt{\nu^4-4}}{2}\right)-4}{\nu^4-4}. \tag{3.20}$$

Proof. We use Lemma 3.15 to determine spectral equation. We compute the following

$$a_{23}(1,\lambda) = -\frac{1}{\lambda\sqrt{-4+\nu^4}}\cosh(\sigma_1(\lambda))+\frac{1}{\lambda\sqrt{-4+\nu^4}}\cosh(\sigma_2(\lambda)),$$

$$
\begin{aligned}
a_{24}(1,\lambda) &= \frac{-\nu^2+\sqrt{-4+\nu^4}}{\sqrt{2}\sqrt{-4+\nu^4}\sqrt{\lambda\left(2\lambda+\nu^2-\sqrt{-4+\nu^4}\right)}}\sinh(\sigma_1(\lambda)) \\
&\quad + \frac{\nu^2+\sqrt{-4+\nu^4}}{\sqrt{2}\sqrt{-4+\nu^4}\sqrt{\lambda\left(2\lambda+\nu^2+\sqrt{-4+\nu^4}\right)}}\sinh(\sigma_2(\lambda)), \\
a_{33}(1,\lambda) &= \frac{2+\lambda\left(\nu^2+\sqrt{-4+\nu^4}\right)}{2\lambda\sqrt{-4+\nu^4}}\cosh(\sigma_1(\lambda)) \\
&\quad + \frac{-2+\lambda\left(-\nu^2+\sqrt{-4+\nu^4}\right)}{2\lambda\sqrt{-4+\nu^4}}\cosh(\sigma_2(\lambda)), \\
a_{34}(1,\lambda) &= \frac{\sqrt{\lambda\left(2\lambda+\nu^2-\sqrt{-4+\nu^4}\right)}}{\sqrt{2}\lambda\sqrt{-4+\nu^4}}\sinh(\sigma_1(\lambda)) \\
&\quad - \frac{\sqrt{\lambda\left(2\lambda+\nu^2+\sqrt{-4+\nu^4}\right)}}{\sqrt{2}\lambda\sqrt{-4+\nu^4}}\sinh(\sigma_2(\lambda)), \\
a_{43}(1,\lambda) &= -\frac{\sqrt{\lambda\left(2\lambda+\nu^2-\sqrt{-4+\nu^4}\right)}}{\sqrt{2}\lambda\sqrt{-4+\nu^4}}\sinh(\sigma_1(\lambda)) \\
&\quad + \frac{\sqrt{\lambda\left(2\lambda+\nu^2+\sqrt{-4+\nu^4}\right)}}{\sqrt{2}\lambda\sqrt{-4+\nu^4}}\sinh(\sigma_2(\lambda)), \\
a_{44}(1,\lambda) &= \frac{-\nu^2+\sqrt{-4+\nu^4}}{2\sqrt{-4+\nu^4}}\cosh(\sigma_1(\lambda)) + \frac{\nu^2+\sqrt{-4+\nu^4}}{2\sqrt{-4+\nu^4}}\cosh(\sigma_1(\lambda)),
\end{aligned}
$$

Now we consider spectral equation $\mathcal{P}(\lambda)=0$, that is

$$
\begin{aligned}
\mathcal{P}(\lambda) = -\frac{2}{-4+\nu^4} + \frac{-2+\nu^4}{-4+\nu^4}\cosh(\sigma_1(\lambda))\cosh(\sigma_2(\lambda)) \\
+\sigma_5\sinh(\sigma_1(\lambda))\sinh(\sigma_2(\lambda)),
\end{aligned}
$$

where

$$
\begin{cases}
\sigma_1(\lambda) = \dfrac{\sqrt{\lambda}\sqrt{2\lambda+\nu^2-\sqrt{-4+\nu^4}}}{\sqrt{2}}, \\
\sigma_2(\lambda) = \dfrac{\sqrt{\lambda}\sqrt{2\lambda+\nu^2+\sqrt{-4+\nu^4}}}{\sqrt{2}}, \\
\sigma_5(\lambda) = -\dfrac{(2\lambda+\nu^2)\sqrt{2\lambda+\nu^2-\sqrt{-4+\nu^4}}\sqrt{2\lambda+\nu^2+\sqrt{-4+\nu^4}}}{2(-4+\nu^4)(1+\lambda^2+\lambda\nu^2)}.
\end{cases}
$$

Note that for $\nu = 0$ our results coincide with those from [63]. To find eigenvalues of $\mathcal{A}_1$ it is sufficient to solve equation

$$-\frac{2}{-4+\nu^4} + \frac{-2+\nu^4}{-4+\nu^4}\cosh(\sigma_1)\cosh(\sigma_2) + \sigma_5 \sinh(\sigma_1)\sinh(\sigma_2) = 0.$$

Proceeding similar as in previous theorem, for sufficiently large $|\lambda|$ we obtain

$$\begin{cases} \sigma_1(\lambda) = \lambda + \frac{1}{4}(\nu^2 - \sqrt{-4+\nu^4}) - \frac{1}{32}(\nu^2 - \sqrt{-4+\nu^4})^2\frac{1}{\lambda} + \varphi_1(\frac{1}{\lambda^2}), \\ \sigma_2(\lambda) = \lambda + \frac{1}{4}(\nu^2 + \sqrt{-4+\nu^4}) - \frac{1}{32}(\nu^2 + \sqrt{-4+\nu^4})^2\frac{1}{\lambda} + \varphi_2(\frac{1}{\lambda^2}), \\ \sigma_5(\lambda) = -\frac{2}{-4+\nu^4} - \frac{1}{4\lambda^2} + \varphi_5(\frac{1}{\lambda^3}), \end{cases}$$

where $\varphi_i(\cdot)$ is analytic function in a neighborhood of 0 with

$$\lim_{|\lambda|\to\infty} \varphi_i\left(\frac{1}{\lambda}\right) = 0.$$

Hence we can define asymptotic approximations of σ_i's as $|\lambda| \to \infty$ in the form

$$\begin{cases} \widetilde{\sigma}_1(\lambda) = \lambda + \frac{1}{4}(\nu^2 - \sqrt{-4+\nu^4}), \\ \widetilde{\sigma}_2(\lambda) = \lambda + \frac{1}{4}(\nu^2 + \sqrt{-4+\nu^4}), \\ \widetilde{\sigma}_5(\lambda) = -\frac{2}{-4+\nu^4}. \end{cases}$$

Now we can consider an approximate equation, in the form

$$\begin{aligned} &-\frac{2}{-4+\nu^4} \\ &+\frac{-2+\nu^4}{-4+\nu^4}\cosh\left(\lambda + \frac{1}{4}(\nu^2 - \sqrt{-4+\nu^4})\right)\cosh\left(\lambda + \frac{1}{4}(\nu^2 - \sqrt{-4+\nu^4})\right) \\ &-\frac{2}{-4+\nu^4}\sinh\left(\lambda + \frac{1}{4}(\nu^2 - \sqrt{-4+\nu^4})\right)\sinh\left(\lambda + \frac{1}{4}(\nu^2 - \sqrt{-4+\nu^4})\right) \\ = {} & 0. \end{aligned}$$

After small calculation we can rewrite it equivalently as

$$\cosh\left(2\lambda + \frac{1}{2}\nu^2\right) = \frac{4 - \nu^4\cosh\left(\frac{\sqrt{\nu^4-4}}{2}\right)}{\nu^4 - 4}, \tag{3.21}$$

for $\nu \in (0,+\infty)\backslash\sqrt{2}$ (for the $\nu = \sqrt{2}$ considerations are similar). Closer analysis of right-hand side of (3.21) shows that for $\nu \in (0,\sqrt{2})$ we have

$$\frac{4-\nu^4\cos\left(\frac{\sqrt{4-\nu^4}}{2}\right)}{\nu^4-4} < -1$$

and for $\nu > \sqrt{2}$

$$\frac{4-\nu^4\cosh\left(\frac{\sqrt{\nu^4-4}}{2}\right)}{\nu^4-4} < -1$$

Thus, eigenvalues for $\nu \neq \sqrt{2}$ are given by

$$\tilde{\lambda} = -\frac{1}{4}\nu^2 + \frac{1}{2}\cosh^{-1}\left(\frac{4-\nu^4\cosh\left(\frac{\sqrt{\nu^4-4}}{2}\right)}{\nu^4-4}\right),$$

or equivalently as

$$\begin{aligned}\tilde{\lambda}^{(1)} &= -\frac{1}{4}\nu^2 + \frac{1}{2}\ln(-y+\sqrt{y^2-1}),\\ \tilde{\lambda}^{(2)} &= -\frac{1}{4}\nu^2 + \frac{1}{2}\ln(-y-\sqrt{y^2-1}),\end{aligned}$$

where y is given by (3.20) and $y > 1$. Therefore, we obtain two families of points

$$\begin{aligned}\tilde{\lambda}^{(1)}_k &= -\frac{1}{4}\nu^2 + \frac{1}{2}\ln(y-\sqrt{y^2-1}) + \frac{2k+1}{2}\pi i,\\ \tilde{\lambda}^{(2)}_k &= -\frac{1}{4}\nu^2 + \frac{1}{2}\ln(y+\sqrt{y^2-1}) + \frac{2k+1}{2}\pi i\end{aligned}$$

(see Fig. 3.5).

□

Remark 3.21. Note that the results for $\gamma = 1$ are qualitatively different than the results for $\gamma^2 > 1$. The main difference is in the asymptotic behavior of σ's. Let us for example consider the following expression

$$\sigma_1\left(\frac{1}{s}\right) - \frac{1}{\gamma}\frac{1}{s} = \frac{4\nu^2\frac{1}{s}\left(-1+\gamma^2\right)+4\gamma^2}{\left(\sqrt{1+\gamma^2+\nu^2 s-\sqrt{-4\gamma^2 s^2+\left(1-\gamma^2+\nu^2 s\right)^2}}+\sqrt{2}\right)} \cdot \frac{1}{\sqrt{2}\gamma\left(-\frac{1}{s}+\frac{1}{s}\gamma^2+\nu^2+\sqrt{-4\gamma^2+\left(\frac{1}{s}-\frac{1}{s}\gamma^2+\nu^2\right)^2}\right)},$$

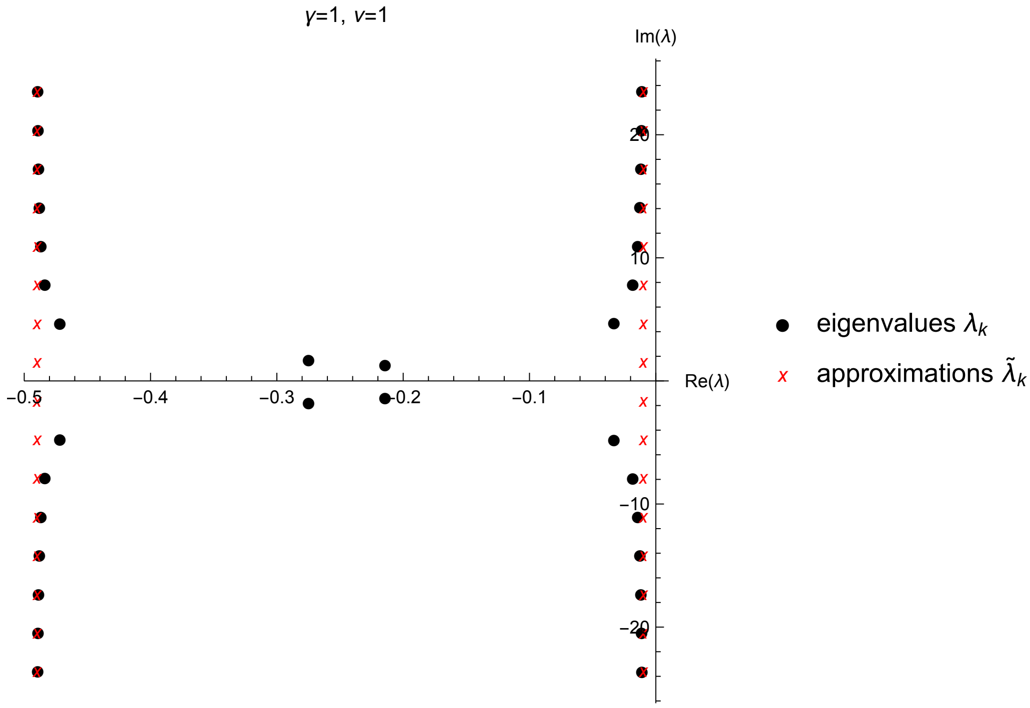

Figure 3.5: Eigenvalues in the case $\gamma = 1$ and $\nu = 1$ (black circles denote eigenvalues λ_k, red crosses denote approximations of eigenvalues $\tilde{\lambda}_k$)

where

$$s = \frac{1}{\lambda}.$$

If we calculate the limit at infinity of this expression, we will get

$$\lim_{|s|\to\infty} \left(\sigma_1 \left(\frac{1}{s} \right) - \frac{1}{\gamma} \frac{1}{s} \right) = \frac{1}{2\gamma} \nu^2,$$

but if we first assume $\gamma = 1$ then terms with the highest order cancel and we obtain significantly different asymptotic behavior, that is

$$\sigma_1 \left(\frac{1}{s} \right) - \frac{1}{\gamma} \frac{1}{s} = \frac{4}{\sqrt{2} \left(\sqrt{2 + \nu^2 s - \sqrt{-4s + \nu^4 s^2}} + \sqrt{2} \right)} \cdot \frac{1}{\left(\nu^2 + \sqrt{-4 + \nu^4} \right)}$$

and

$$\lim_{|s|\to\infty} \left(\sigma_1 \left(\frac{1}{s} \right) - \frac{1}{\gamma} \frac{1}{s} \right) = \frac{1}{4} \left(\nu^2 - \sqrt{-4 + \nu^4} \right).$$

To complete considerations with $\gamma^2 \geq 1$, we examine the limit case $\nu = \infty$. Remind that for any $\nu > 0$, we have

$$\begin{cases} \ddot{w}(x,t) - w''(x,t) - \xi'(x,t) = -u(t)(r+x), \\ \ddot{\xi}(x,t) - \gamma^2\xi''(x,t) + w'(x,t) + \xi(x,t) + \nu^2\dot{\xi}(x,t) = u(t), \end{cases}$$

for $x \in (0,1)$ and $t > 0$ with boundary conditions (2.4). Dividing the second of these equations by ν^2 and passing with $\nu \to \infty$, we obtain

$$\begin{cases} \ddot{w}(x,t) - w''(x,t) - \xi'(x,t) = -u(t)(r+x), \\ \dot{\xi}(x,t) = 0, \end{cases} \tag{3.22}$$

for $x \in (0,1)$ and $t > 0$ with boundary conditions (2.4). Now we rewrite system (3.22) in operator equation form

$$\begin{pmatrix} \dot{w} \\ \dot{\xi} \\ \ddot{w} \\ \ddot{\xi} \end{pmatrix} = \mathcal{A}_\infty \begin{pmatrix} w \\ \xi \\ \dot{w} \\ \dot{\xi} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -r-x \\ 0 \end{pmatrix} u(t),$$

where

$$\mathcal{A}_\infty \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_3 \\ 0 \\ y_1'' + y_2' \\ 0 \end{pmatrix},$$

with $D(\mathcal{A}_\infty) = D(\mathcal{A}_1)$.

We arrive at

Theorem 3.22. *For infinite value of a damping coefficient $\nu = \infty$ eigenvalues are $\lambda \in \left\{\frac{2k+1}{2}\pi i : k \in \mathbb{Z}\right\} \cup \{0\}$, thus the system is unstable.*

We skip the proof. Note that the behavior of the system in the case of $\nu = \infty$ is the same in both $\gamma = 1$ and $\gamma^2 > 1$ situations.

The form of the spectrum in all considered cases allows us to formulate the following corollary.

Corollary 3.23. For any value of a damping constant $0 < \nu < \infty$ spectrum $\sigma(\mathcal{A}_1)$ consists of isolated single eigenvalues of $\mathcal{A}_1$.

Now we can observe that the semigroup satisfies spectrum determined growth condition.

Corollary 3.24. Taking into account information from Section 3.3, Corollary 3.19 and Theorem 1.60 we can observe that $\mathcal{A}_1$ is a Riesz-spectral operator (see Definition 1.61), which means that the spectrum determined growth condition is satisfied (see Theorem 1.62.d)).

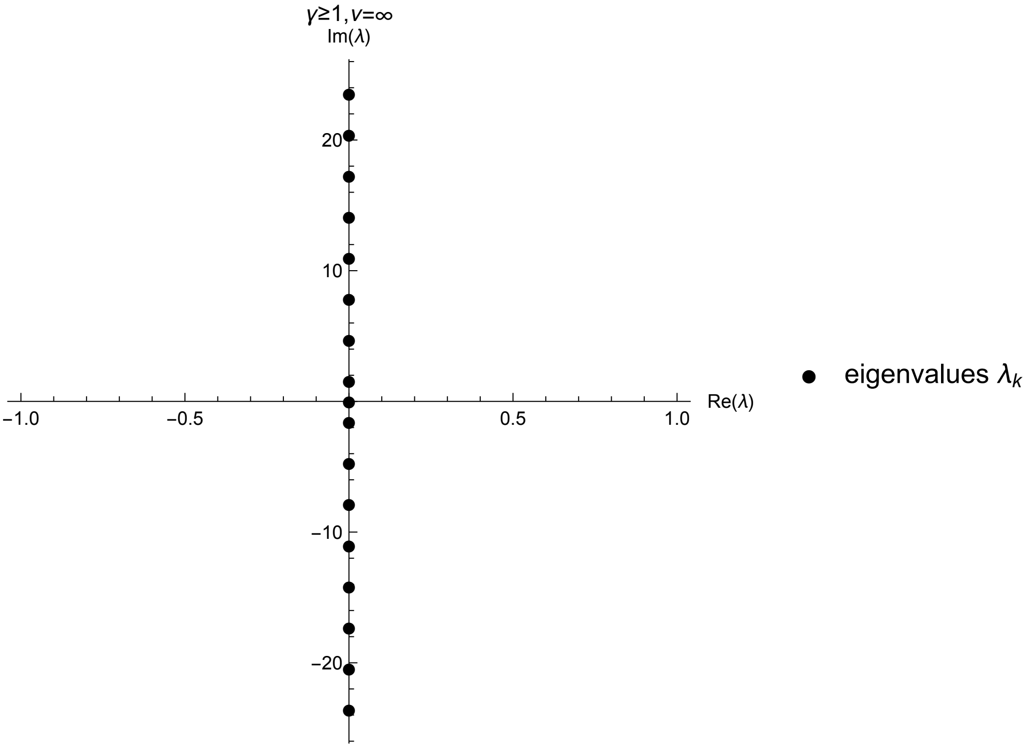

Figure 3.6: Eigenvalues in the limit case $\gamma \geq 1$ and $\nu = \infty$ (black circles denote eigenvalues λ_k)

3.7 Optimal Decay Rate Analysis

In a typical situation (in both cases of finite- and infinite-dimensional systems) we can observe the following behavior of the system: increasing of a damping coefficient (e.g. friction) causes faster energy dissipation for small values (underdamping), then we reach optimal damping coefficient, when the energy dissipates in the fastest way, and for larger damping coefficients than the optimal the energy dissipation is slower (overdamping). In Fig. 3.7 we present plots of the stability margin of damped spring-mass system ($\ddot{x} + d\dot{x} + x = 0$) and damped string ($\ddot{u} - u'' + d\dot{u} = 0$). In our case we will see a similar situation, although the graph will have no wedges.

Finding supremum of the spectrum of operator $\mathcal{A}_1$ gives us information about growth bound of a semigroup and allows us to calculate stability margin. In some cases we are able to find optimal damping coefficient as defined below.

Definition 3.25. The damping coefficient ν_0 will be called *optimal*, if system (2.8) with $\nu = \nu_0$ admits the fastest possible energy dissipation, that is if ν_0

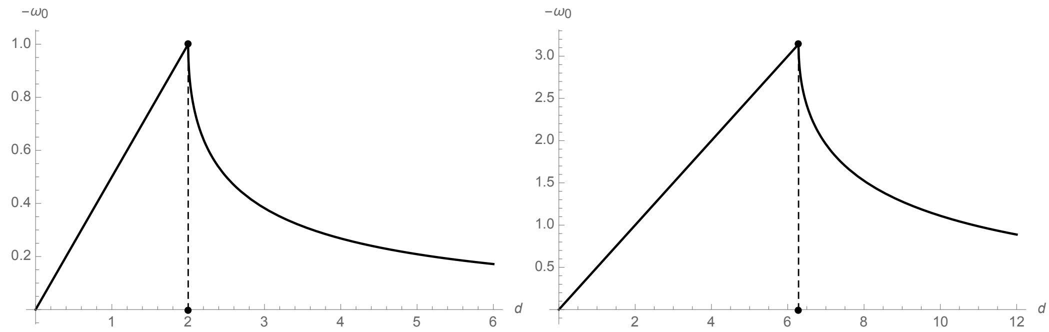

Figure 3.7: Stability margin of damped spring-mass system (left) and damped string (right)

maximizes the stability margin of the system.

In the case of $\gamma^2 > 1$ there is no reason to consider finding the optimal damping coefficient, even on selected subspaces, as one family stays on the imaginary axis and another one escapes to left infinity with increase of the damping coefficient (see Fig. 3.8).

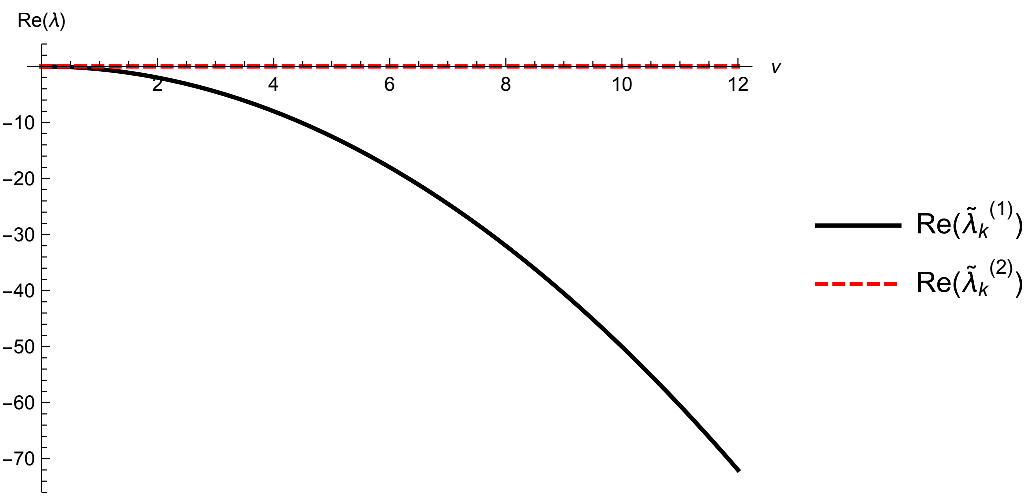

Figure 3.8: Plots of $\mathrm{Re}\left(\tilde{\lambda}_k^{(1)}\right)$ (solid line) and $\mathrm{Re}\left(\tilde{\lambda}_k^{(2)}\right)$ (dashed line) for $\gamma^2 > 1$

Corollary 3.26. Including damping effect to a Timoshenko beam model caused a partial exponential stability, because only one family, $\{\lambda_k^{(1)}\}_{k=k_0}^{\infty}$, is located on left side of complex plane, while second family, $\{\lambda_k^{(2)}\}_{k=k_0}^{\infty}$, is still close to imaginary axis $\mathrm{Re}\,\lambda = 0$.

Remark 3.27. Corollary 3.24 shows that the system satisfies spectrum determined growth condition. We proved that the imaginary axis is the asymptote of the spectrum of $\mathcal{A}$ for $\gamma^2 > 1$ (Theorem 3.17), this implies that the system is not exponentially stable in this case.

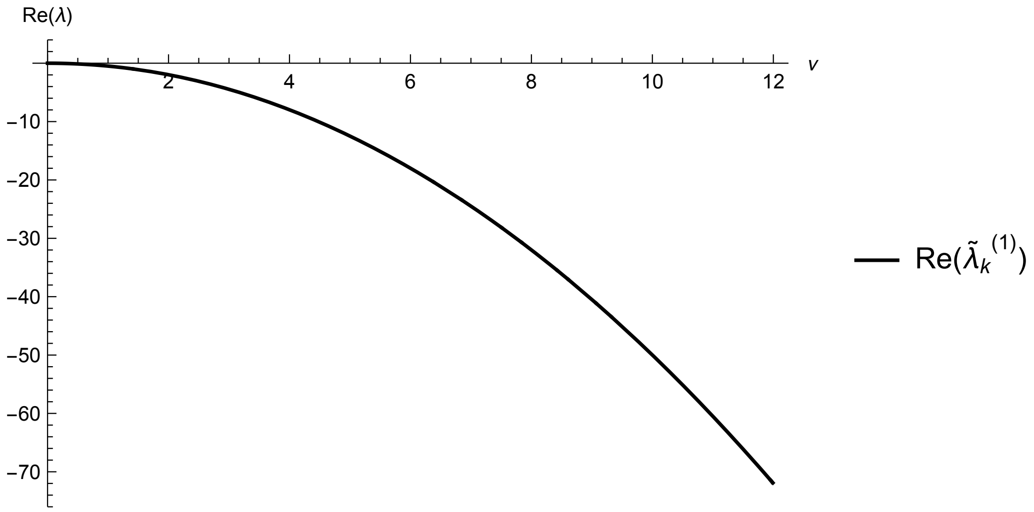

Figure 3.9: Plot of $\mathrm{Re}\left(\tilde{\lambda}_k^{(1)}\right)$ in the case $\gamma = 1$

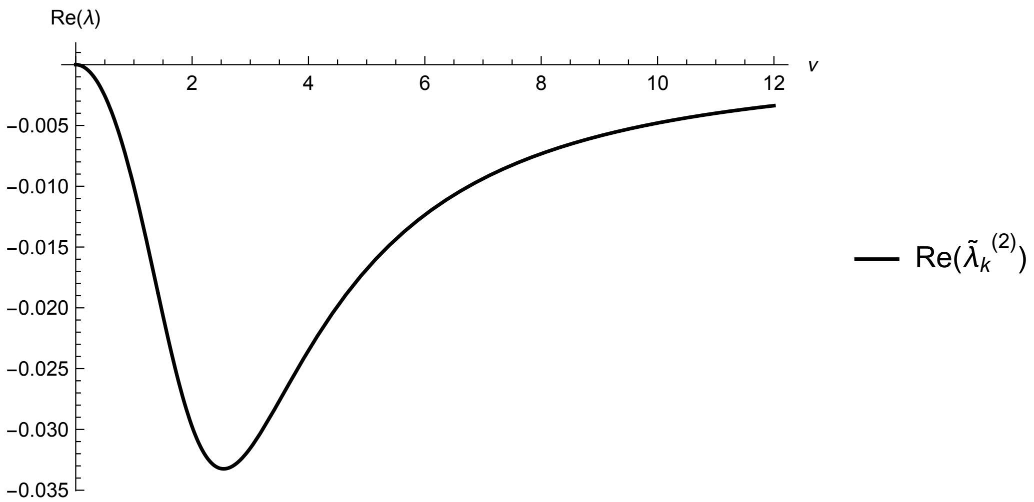

Figure 3.10: Plot of $\mathrm{Re}\left(\tilde{\lambda}_k^{(2)}\right)$ in the case $\gamma = 1$

In the case of $\gamma = 1$ we prove that there exists the optimal damping coefficient for which energy dissipates in the fastest way. Increasing of a damping coefficient ν caused that the first family is moving away from imaginary axis (see Fig. 3.9), while for the second family there exists the optimal damping coefficient (see Fig. 3.10). Increasing ν above the optimal value causes slower energy dissipation (overdamping).

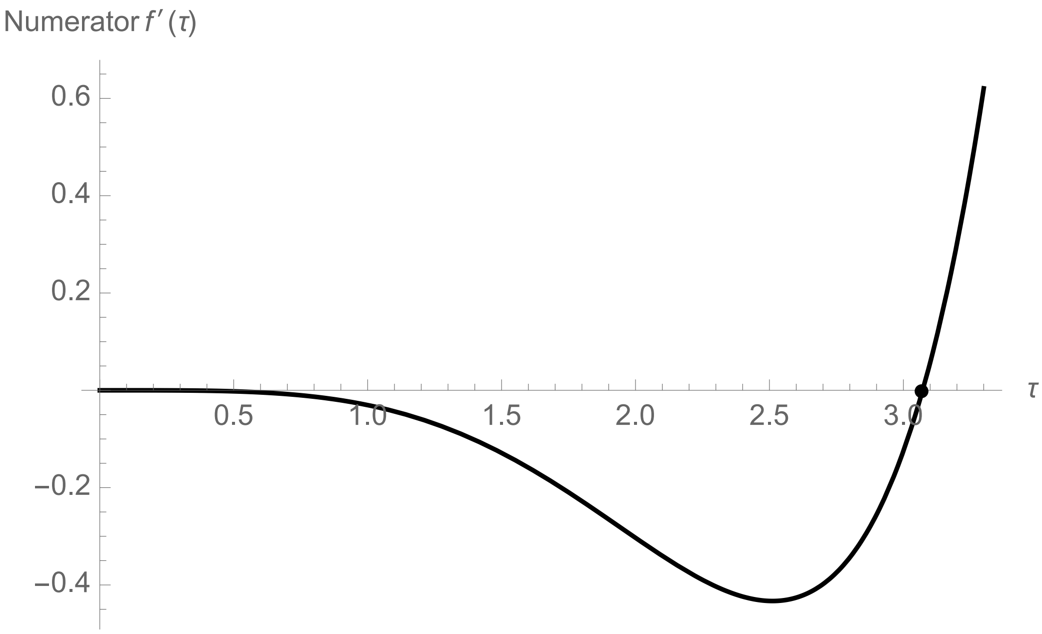

Figure 3.11: Plot of a numerator part of $f'(\tau)$ with marked solution of a $f'(\tau) = 0$

We are able to determine optimal decay ratio of the system in question.

Corollary 3.28. The optimal decay ratio of damped slowly rotating Timoshenko beam (under assumption $\gamma = 1$) is $\omega_0 = -0.033241639124977735136$ (for $\nu_{\text{opt}} = 2.54189087636624306026$).

Proof. From Theorem 3.20 we can observe that

$$\operatorname{Re}\left(\tilde{\lambda}_k^{(1)}\right) < \operatorname{Re}\left(\tilde{\lambda}_k^{(2)}\right),$$

i.e.

$$-\frac{1}{4}\nu^2 + \frac{1}{2}\ln(y - \sqrt{y^2 - 1}) < -\frac{1}{4}\nu^2 + \frac{1}{2}\ln(y + \sqrt{y^2 - 1}).$$

Let us introduce

$$f(\nu) = \operatorname{Re}\left(\tilde{\lambda}_k^{(2)}\right) = -\frac{1}{4}\nu^2 + \frac{1}{2}\ln\left(y + \sqrt{y^2 - 1}\right),$$

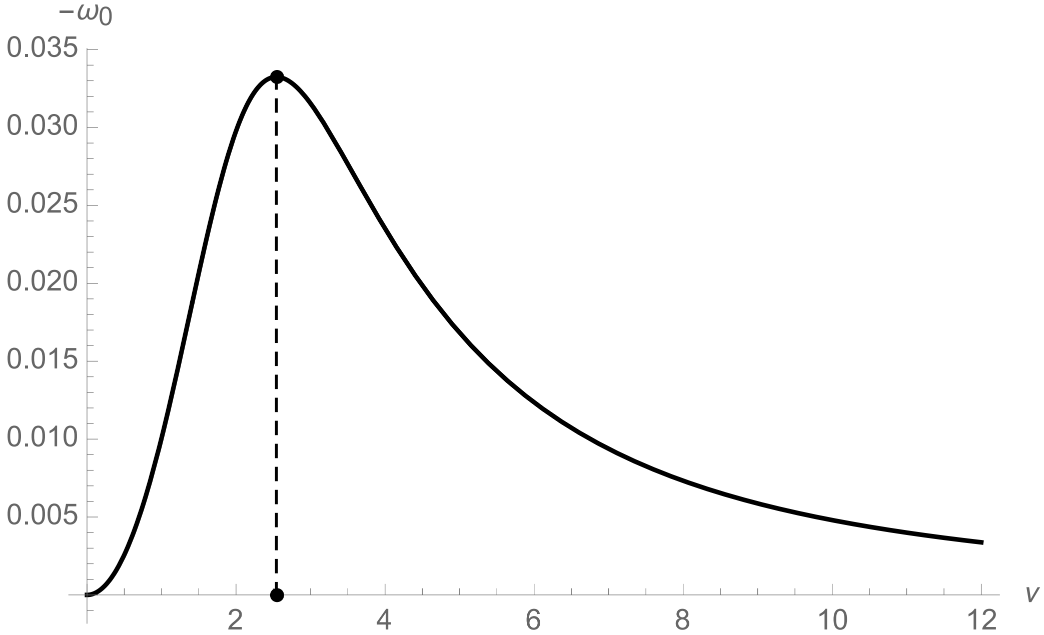

Figure 3.12: Stability margin with an optimal damping coefficient marked in the case $\gamma = 1$

where y is given by (3.20). We need to find minimum of $f(\nu)$, to this end we solve equation $f'(\nu) = 0$. At the beginning we substitute $\tau = \frac{\sqrt{\nu^4-4}}{2} > 0$ and obtain

$$\begin{aligned} f(\tau) = &-\frac{1}{2}\sqrt{\tau^2+1} \\ &+ \frac{1}{2}\ln\left(\frac{(\tau^2+1)\cosh(\tau)-1}{\tau^2} + \sqrt{\left(\frac{(\tau^2+1)\cosh(\tau)-1}{\tau^2}\right)^2 - 1}\right). \end{aligned}$$

Thus, derivative with respect to τ is given by

$$f'(\tau) = \frac{f_n'(\tau)}{f_d'(\tau)},$$

where

$$\begin{aligned} f_n'(\tau) = &\left(\tau^3+\tau\right)\sinh(\tau) - \tau^2\sqrt{(\cosh(\tau)-1)\left(\tau^2+(\tau^2+1)\cosh(\tau)-1\right)} \\ &- 2\cosh(\tau) + 2, \\ f_d'(\tau) = &\, 2\tau^3\sqrt{\frac{\left((\tau^2+1)\cosh(\tau)-1\right)^2}{\tau^4} - 1}. \end{aligned}$$

Note that the solution[1] of a numerator part of a $f'(\tau) = 0$ is approximately equal to $\tau_{opt} = 3.07193850360174816424$ (see Fig. 3.11), leading to $\nu_{opt} = \sqrt[4]{4\tau_{opt}^2 + 4} \approx 2.54189087636624306026$.

Observe that the minimum of a real part of $\lambda^{(2)}$ family is -0.033241639 12497735136 for $\nu = \nu_{opt}$. Thus, by Definition 3.25, we see that the optimal damping coefficient is ν_{opt}, resulting in the optimal decay rate, and for $\nu > \nu_{opt}$ we observe an overdamping effect.

$\square$

Comparing those results according to Definition 3.2, we can plot stability margin dependence on the damping coefficient ν (see Fig 3.12).

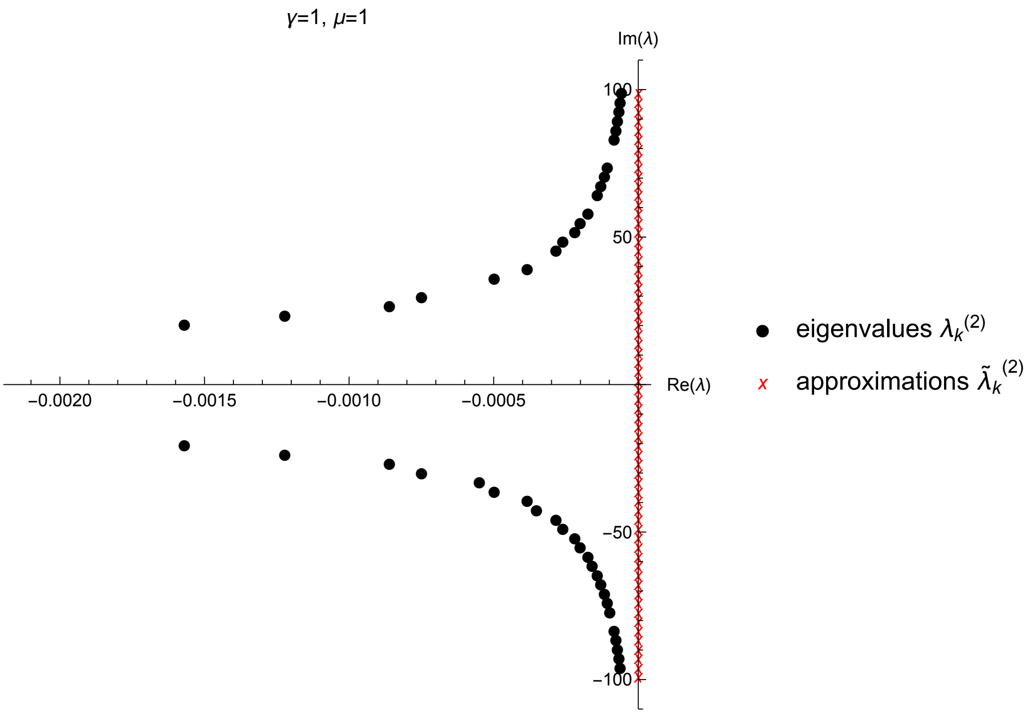

Figure 3.13: Eigenvalues in the case $\gamma = 1$ and $\mu = 1$ (only family $\lambda_k^{(2)}$ shown)

[1]Numerical analysis conducted using Wolfram Mathematica 10.

3.8 Comparison with Other Damping Systems

Analysis of spectrum of operator $\mathcal{A}_1$ in the case described in Section 3.6 and 3.7 shows that for physical parameter $\gamma^2 > 1$ one family of eigenvalues is asymptotically close to imaginary axis, thus no stability margin may be expected. Therefore, in the further analysis we will omit cases where $\gamma^2 > 1$ and we will analyze only those with $\gamma = 1$.

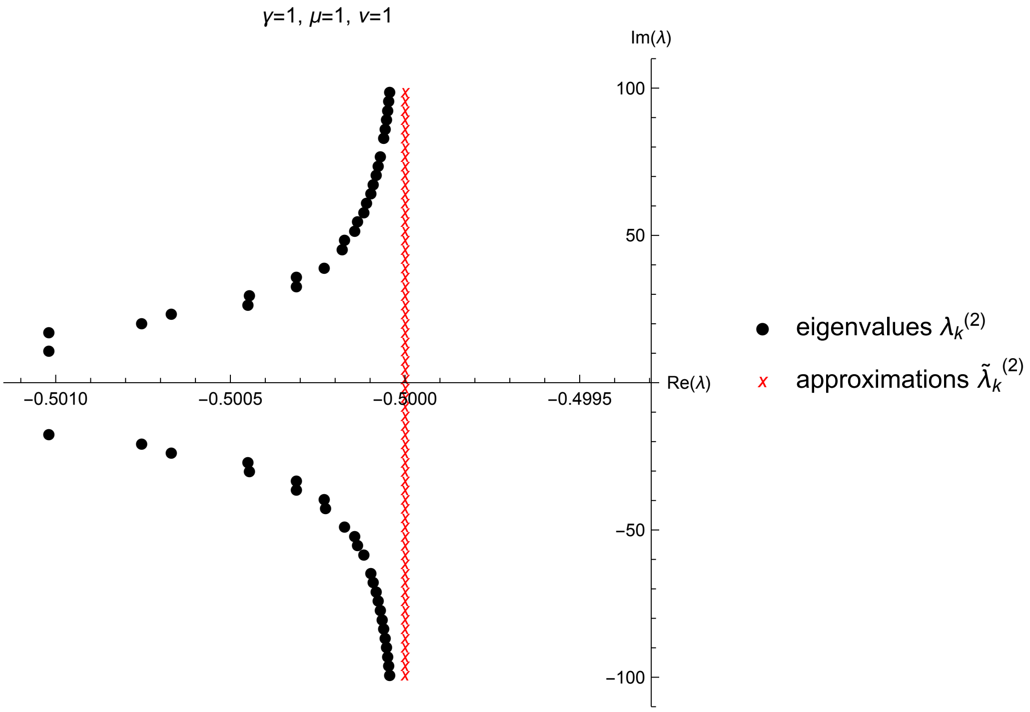

Figure 3.14: Eigenvalues in the case $\gamma = 1$ and $\mu = 1$, $\nu = 1$ (only family $\lambda_k^{(2)}$ shown)

In the following theorem we find approximations of eigenvalues of the operator A_2, i.e. the operator of motion of Timoshenko beam with damping operator B_2 defined by (2.10).

Theorem 3.29. *Let $\gamma = 1$. For any value of a damping constant $0 < \mu < \infty$ the eigenvalues of operator $\mathcal{A}_2$ form two asymptotic families $\lambda_k^{(1)} = -\nu^2 \left(\frac{2k+1}{2}\pi\right)^2 + \varepsilon_k^{(1)}$ and $\lambda_k^{(2)} = \frac{2k+1}{2}\pi i + \varepsilon_k^{(2)}$, where $\lim\limits_{k\to\infty} \varepsilon_k^{(i)} = 0$ (see Fig. 3.13).*

Corollary 3.30. From Theorem 3.29, we can observe that the imaginary axis is an asymptote of the spectrum of $\mathcal{A}_2$. In this case stability margin is equal to 0, so there is no reason to consider optimal stability margin.

To complete our consideration with $\gamma = 1$, we examine the case with additive combination of damping operators (2.9) and (2.10), i.e. damping operator (2.11).

Theorem 3.31. *Let $\gamma = 1$. For any value of a damping constant $0 < \mu, \nu < \infty$ the eigenvalues of operator $\mathcal{A}_3$ consist of two asymptotic families $\lambda_k^{(1)} = -\mu^2 \left(\frac{2k+1}{2}\pi\right)^2 + \varepsilon_k^{(1)}$ and $\lambda_k^{(2)} = -\frac{1}{2}\nu^2 + \frac{2k+1}{2}\pi i + \varepsilon_k^{(2)}$, where $\lim\limits_{k\to\infty} \varepsilon_k^{(i)} = 0$ (see Fig. 3.14).*

Corollary 3.32. From Theorem 3.31, we can see that asymptote of the spectrum of $\mathcal{A}_3$ is given by $\min\left\{\mu^2\left(\frac{2k+1}{2}\pi\right)^2, \frac{1}{2}\nu^2\right\}$. There is no reason of looking for optimal stability margin here because stability margin of the system goes to infinity as damping coefficients go to infinity.

Chapter 4

Observability Analysis

Another important concept in control theory is observability. Observability is a property of the control system that allows to check whether it is possible to determine the internal state of an object based on the knowledge of input (control) and output (observation).

In this chapter we consider the problem of exact observability of a general class of distributed parameter systems in Hilbert spaces. We show that under some conditions on asymptotic behavior of the spectrum of the differential operator the system is not exactly observable in default topologies, and we find a stronger topology for state observation for which the system becomes exactly observable. We illustrate this result with a vibrating clamped–free Timoshenko beam model.

Main results of this chapter were published in [54].

4.1 Various Concepts of Observablity

There can be distinguished three important observability notions mostly used in literature. In this section we introduce and briefly discuss them.

Let $A : D(A) \subset H \to H$ is unbounded, positive definite linear operator. Consider differential equation with observation of the form

$$\begin{cases} \dot{z}(t) = \mathcal{A}z(t), \\ y(t) = \mathcal{C}z(t), \end{cases} \tag{4.1}$$

where $z(t) = (v(t), w(t))^T$, $v, w \in H$, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\mathcal{T}(t)$ given in the product space[1] $\mathcal{H} = D\left(A^{\frac{1}{2}}\right) \times H$ and defined

[1]Operator $A^{\frac{1}{2}}$ is defined as in Section 2.2

by

$$\mathcal{A}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix},$$

where $I : D\left(A^{\frac{1}{2}}\right) \to H$ is embedding operator, with domain $D(\mathcal{A}) = D(A) \times D\left(A^{\frac{1}{2}}\right) \subset \mathcal{H}$, and $\mathcal{C} : \mathcal{H} \to \mathbb{C}$ is a linear (unbounded) observation operator.

In addition, we assume that observation operator is admissible, in the following sense

Definition 4.1. The operator $\mathcal{C}$ is called an *admissible operator* for semigroup $\mathcal{T}(t)$ if, for some $T > 0$ (and hence for all $T > 0$), there exists a constant $K \geq 0$ such that

$$\int_0^T |\mathcal{C}\mathcal{T}(t)z_0|^2 \, dt \leq K^2 \|z_0\|^2 \;\; \forall z_0 \in D(\mathcal{A}).$$

We use the following classical notions of observability for unbounded operators.

Definition 4.2. Let $\mathcal{K} : H \to \mathcal{Y}$ be the output operator

$$z_0 \mapsto \mathcal{K}z_0 = \mathcal{C}\mathcal{T}(t)z_0,$$

where $\mathcal{Y}$ is a Hilbert space of time-dependent functions on the interval $(0, T)$. The system (4.1) (or the pair $(\mathcal{A}, \mathcal{C})$) is said to be *approximately observable in time* T (or *observable in time* T) if $\ker \mathcal{K} = \{0\}$, *final state observable in time* T if

$$\|\mathcal{K}z_0\|_{\mathcal{Y}}^2 \geq \kappa^2 \|\mathcal{T}(T)z_0\|_H^2 \;\; \forall z_0 \in H,$$

for some constant $\kappa > 0$ and $\mathcal{Y} - H$ *exactly observable in time* T (or $\mathcal{Y} - H$ *continuously observable in time* T) if

$$\|\mathcal{K}z_0\|_{\mathcal{Y}}^2 \geq \kappa^2 \|z_0\|_H^2 \;\; \forall z_0 \in H, \tag{4.2}$$

for some constant $\kappa > 0$.

For finite-dimensional systems there is one concept of observability (the concepts from Definition 4.2 are equivalent), which is independent of time. In this case, the observability can be checked by *Kalman rank condition* (see [60]).

Theorem 4.3. *The pair* $(\mathcal{A}, \mathcal{C})$ *is observable if and only if*

$$\operatorname{rank}\begin{bmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \vdots \\ \mathcal{C}\mathcal{A}^{n-1} \end{bmatrix} = n.$$

The concept of observability becomes much more complicated if we pass from finite- to infinite-dimensional state space. Briefly speaking approximate observability guarantees possibility of reconstruction of initial state of the system and, in consequence, of a whole trajectory, using knowledge of the output. Exact observability and final state observability mean that we can find the initial and final state respectively from the given output with infinitesimal precision, that is for any convergent sequence of pairwise different initial (final) states the resulting outputs are convergent and pairwise different as well. Of course exact observability implies both approximate observability and final state observability, but not the other way around [60]. In this chapter we focus on analysis of exact observability notion.

The exact observability depends heavily on the choice of topology in the space as opposed to approximate observability [44]. We will prove that in default topologies setting the considered system (4.1) is not exactly observable under some specific conditions. There are two possible methods to overcome this problem for further studies of exact observability of system (4.1), we can use the weaker topology on the right-hand side of the inequality (4.2) or stronger topology on the left-hand side of the inequality (4.2). Usually the first approach is taken (see e.g. [60]) using the domain $D(\mathcal{A})$ of operator $\mathcal{A}$ or of its power $D(\mathcal{A}^n)$ as a space of initial states to be observed, that is changing the right-hand side of the inequality (4.2). We are going to use the second approach in our research, i.e. find stronger topology on the left-hand side of the inequality (4.2). Our investigation will be based on the moment problem approach.

The moment problem method is known as one of the most powerful tools in modern control systems theory. Starting from the works of N. Krasovskii [36], D. Russell [47], H. Fattorini [15] and others, many papers are devoted to research of the controllability problem using the moment problem approach. The main advantage is the use of profound results of complex functions theory, allowing to solve the problems that seemed for many years to be very hard to analyze. One of such problems is exact observability problem for infinite dimensional systems, that is the exact description of observable states. It turned out that in many cases the solution of this problem can be obtained using the results of Riesz basis properties of exponential families (see [3] and references within). Another example is the problem of time-optimality for systems of arbitrary dimensions, that was solved [30, 31] using classical Markov moment problem approach (M. G. Krein, A. A. Nudelman [37]).

4.2 Exact Observability Conditions

In this section, we present general exact observability analysis of system (4.1). At the beginning, we note that system (4.1) is a general form of second order system its observability results will depend strongly on the spectral analysis of the operators in question: location of the spectrum and properties of eigensystems. Thus we formulate some specific assumptions we impose on the system in question:

(A1) The operator $\mathcal{A}$ has an orthogonal complete sequence of eigenelements $\{Y_k\}_{k\in\mathbb{Z}}$ with corresponding eigenvalues $\mu_k = \pm i\sqrt{\lambda_k}$, where[2] $\lambda_k \asymp k^2$ denotes an increasing sequence of (real, positive) eigenvalues of operator A.

(A2) There exists an increasing sequence $\{k_n\}_{n\in\mathbb{Z}}$ of indices such that eigenvalues μ_{k_n} and μ_{k_n-1} are approaching each other with a certain speed, $|\mu_{k_n} - \mu_{k_n-1}| \asymp \frac{1}{|k_n|}$.

(A3) For some $T_0 > 0$ the system

$$\left\{e^{\mu_k t}\right\}_{k\in\mathbb{Z}\setminus\{k_n\}} \cup \left\{\frac{e^{\mu_k t} - e^{\mu_{k-1} t}}{\mu_k - \mu_{k-1}}\right\}_{k\in\{k_n\}} \tag{4.3}$$

is a Riesz basis for $L^2(0, T_0)$.

(A4) $|C_k| \asymp 1$, where $C_k := \mathcal{C}Y_k$, when the sequence of norms of eigenvectors Y_k is almost k-normalized, that is asymptotically bounded from below and above by $|k|$, as $|k| \to \infty$, i.e. $\|Y_k\|_{\mathcal{H}} \asymp |k|$.

We start our considerations with a simple remark concerning approximate observability notion.

Proposition 4.4. The system (4.1) under conditions (A1) and (A3) is approximately observable for $T \geq T_0$ and is not approximately observable for $T < T_0$.

Corollary 4.5. Since exact observability implies approximate observability and lack of approximate observability implies lack of exact observality, then exact observability phenomenon of system (4.1) can appear only for intervals with final times $T \geq T_0$.

[2] $\alpha_k \asymp \beta_k$ iff $|\alpha_k| \leq C\,|\beta_k|$ and $|\beta_k| \leq C\,|\alpha_k|$ for some C and for sufficiently large k, see [17, p. 442]

Next, we proceed with exact observability notion analysis, namely we show lack of exact observability in the default topologies setting.

Theorem 4.6. *Assume that the conditions (A1) and (A4) hold. Then the system* (4.1) *is not* $L^2(0,T) - \mathcal{H}$ *exactly observable in any time* $T > 0$.

Proof. Consider sequence $\{Y_k\}_{k\in\mathbb{Z}}$ to be observed. We have

$$\mathcal{K}Y_k = \mathcal{C}\mathcal{T}(t)Y_k = \mathcal{C}e^{\mu_k t}Y_k = e^{\mu_k t}C_k$$

for any $t > 0$. Thus,

$$\|\mathcal{K}Y_k\|^2_{L^2(0,T)} = \int_0^T \left|e^{\mu_k t}C_k\right|^2 dt = |C_k|^2 T \asymp 1,$$

because $\mu_k \in i\mathbb{R}$ (see (A1)) and $|C_k| \asymp 1$ (see (A4)). On the other hand, by (A4),

$$\|Y_k\|_{\mathcal{H}} \asymp |k|.$$

Then

$$\frac{\|\mathcal{K}Y_k\|_{L^2(0,T)}}{\|Y_k\|_{\mathcal{H}}} \to 0 \text{ as } |k| \to \infty.$$

Thus, inequality (4.2) cannot hold, hence the system (4.1) is not $L^2(0,T) - \mathcal{H}$ exactly observable in any time $T > 0$, which finishes the proof. $\square$

Now, we are ready to proceed with the main result of the section. We prove that for $T \geq T_0$ the condtitions (A1)–(A4) imply exact observability in time T of system (4.1) after strengthening one of the topologies in question, namely replacing $\mathcal{Y} = L^2(0,T)$ by $\mathcal{Y} = H^2(0,T)$.

Theorem 4.7. *Assume that the conditions (A1)–(A4) are satisfied. Let* $T \geq T_0$. *Then system* (4.1) *is* $H^2(0,T) - \mathcal{H}$ *exactly observable in time* T.

Proof. Let $T = T_0$, for $T > T_0$ exact observability will be obvious. At the beginning we will estimate the norm of the left-hand side of inequality (4.2), $\|\mathcal{K}z_0\|^2_{H^2(0,T_0)}$. In order to do this, we present state vector in eigenvector space (A1) and then we decompose it in Riesz basis (4.3) from (A3). The arbitrary state $z_0 \in \mathcal{H}$ may be written in the normalized eigenvector space in the following form

$$z_0 = \sum_{k\in\mathbb{Z}} \alpha_k \frac{Y_k}{\|Y_k\|}, \tag{4.4}$$

where $\alpha_k = \left\langle z_0, \frac{Y_k}{\|Y_k\|} \right\rangle \in \ell^2$ and Y_k is an eigenvector from (A1). Then, after using the form of the operator $\mathcal{K}$, we obtain

$$\mathcal{K}z_0 = \mathcal{C}\mathcal{T}(t)z_0 = \sum_{k\in\mathbb{Z}} \alpha_k e^{\mu_k t} \frac{C_k}{\|Y_k\|}.$$

Now, we decompose it in Riesz basis (4.3)

$$\sum_{k\in\mathbb{Z}} \alpha_k e^{\mu_k t} \frac{C_k}{\|Y_k\|} = \sum_{k\in\mathbb{Z}\setminus\{k_n\}} \beta_k e^{\mu_k t} + \sum_{k\in\{k_n\}} \gamma_k \frac{e^{\mu_k t} - e^{\mu_{k-1} t}}{\mu_k - \mu_{k-1}}, \tag{4.5}$$

where coefficients in Riesz basis are given by the following formulas

$$\beta_k = \frac{C_k}{\|Y_k\|}\alpha_k \text{ for } k \neq k_n - 1, k_n$$

and

$$\begin{aligned} \beta_{k_n-1} &= \frac{C_{k_n}}{\|Y_{k_n}\|}\alpha_{k_n} + \frac{C_{k_n-1}}{\|Y_{k_n-1}\|}\alpha_{k_n-1}, \\ \gamma_{k_n} &= (\mu_{k_n} - \mu_{k_n-1}) \frac{C_{k_n-1}}{\|Y_{k_n-1}\|}\alpha_{k_n-1} \end{aligned}$$

for remaining cases. Now, we proceed with estimation of left-hand side of the inequality (4.2). We change the topology of the space and use stronger norm, i.e. $H^2(0, T_0)$ norm. Hence, the norm of $\mathcal{K}z_0$ is calculated as

$$\|\mathcal{C}\mathcal{T}(t)z_0\|^2_{H^2(0,T_0)} = \|\mathcal{C}\mathcal{T}(t)z_0\|^2_{L^2(0,T_0)} + \left\| \frac{d^2}{dt^2} \left(\mathcal{C}\mathcal{T}(t)z_0\right) \right\|^2_{L^2(0,T_0)}. \tag{4.6}$$

In general if family $\{\varphi_k\}$ is a Riesz basis then there exist constants $m, M > 0$ such that for any sequence $(x_k) \in \ell^2$ one has

$$m \sum |x_k|^2 \leq \left\| \sum x_k \varphi_k \right\|^2 \leq M \sum |x_k|^2 \tag{4.7}$$

(see Remark 1.59). Using (4.5) and (4.7), the estimation for the first term of (4.6) is given by

$$\|\mathcal{C}\mathcal{T}(t)z_0\|^2_{L^2(0,T_0)} \geq m \left(\sum_{k\in\mathbb{Z}\setminus\{k_n\}} |\beta_k|^2 + \sum_{k\in\{k_n\}} |\gamma_k|^2 \right). \tag{4.8}$$

We present necessary calculations for the second term of (4.6), namely

$$\begin{aligned}\frac{d^2}{dt^2}\left(\mathcal{C}\mathcal{T}(t)z_0\right) &= \sum_{k\in\mathbb{Z}\setminus\{k_n\}} \mu_k^2\beta_k e^{\mu_k t} + \sum_{k\in\{k_n\}} \gamma_k \frac{\mu_k^2 e^{\mu_k t} - \mu_{k-1}^2 e^{\mu_{k-1} t}}{\mu_k - \mu_{k-1}} \\ &= \sum_{k\in\mathbb{Z}\setminus\{k_n\}} \mu_k^2\beta_k e^{\mu_k t} + \sum_{k\in\{k_n\}} \left(\mu_k + \mu_{k-1}\right)\gamma_k e^{\mu_k t} \\ &\quad + \sum_{k\in\{k_n\}} \mu_{k-1}^2\gamma_k \frac{e^{\mu_k t} - e^{\mu_{k-1} t}}{\mu_k - \mu_{k-1}}.\end{aligned}$$

After rearranging the terms, we obtain

$$\begin{aligned}\frac{d^2}{dt^2}(\mathcal{C}\mathcal{T}(t)z_0) = &\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} \mu_k^2\beta_k e^{\mu_k t} \\ &+ \sum_{k\in\{k_n-1\}} \left(\mu_k^2\beta_k + \left(\mu_{k+1} + \mu_k\right)\gamma_{k+1}\right) e^{\mu_k t} \\ &+ \sum_{k\in\{k_n\}} \mu_k^2\gamma_k \frac{e^{\mu_k t} - e^{\mu_{k-1} t}}{\mu_k - \mu_{k-1}}.\end{aligned}$$

Again, using the properties of Riesz basis (4.7), we obtain

$$\begin{aligned}\left\|\frac{d^2}{dt^2}(\mathcal{C}\mathcal{T}(t)z_0)\right\|_{L^2}^2 \geq m &\left(\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} \left|\mu_k^2\beta_k\right|^2\right. \\ &+ \sum_{k\in\{k_n-1\}} \left|\mu_k^2\beta_k + \left(\mu_{k+1} + \mu_k\right)\gamma_{k+1}\right|^2 \\ &\left.+ \sum_{k\in\{k_n\}} \left|\mu_k^2\gamma_k\right|^2\right).\end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) we obtain estimation from below for (4.6), namely

$$\begin{aligned}\|\mathcal{C}\mathcal{T}(t)z_0\|_{H^2}^2 \geq m &\left(\sum_{k\in\mathbb{Z}\setminus\{k_n\}} |\beta_k|^2 + \sum_{k\in\{k_n\}} |\gamma_k|^2\right) \\ &+ m\left(\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} \left|\mu_k^2\beta_k\right|^2\right. \\ &\left.+ \sum_{k\in\{k_n-1\}} \left|\mu_k^2\beta_k + \left(\mu_{k+1} + \mu_k\right)\gamma_{k+1}\right|^2 + \sum_{k\in\{k_n\}} \left|\mu_k^2\gamma_k\right|^2\right),\end{aligned}$$

in particular

$$\|\mathcal{C}\mathcal{T}(t)z_0\|_{H^2}^2 \geq m\left(\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}\left|\mu_k^2\beta_k\right|^2\right.$$
$$+\sum_{k\in\{k_n-1\}}\left|\mu_k^2\beta_k+(\mu_{k+1}+\mu_k)\,\gamma_{k+1}\right|^2 \tag{4.10}$$
$$\left.+\sum_{k\in\{k_n\}}\left|\mu_k^2\gamma_k\right|^2\right).$$

Now, we will estimate the right-hand side of the inequality (4.2). Let us consider the norm of the arbitrary state $z_0 \in \mathcal{H}$. It is obvious (due to normalized expansion (4.4)) that the norm of the state z_0 is given by

$$\|z_0\|_{\mathcal{H}}^2 = \sum_{k\in\mathbb{Z}} |\alpha_k|^2.$$

From (4.5) we can derive the formulas for coefficients α_k as

$$\alpha_k = \frac{\|Y_k\|}{C_k}\beta_k \text{ for } k \neq k_n-1, k_n$$

and

$$\alpha_{k_n-1} = \frac{\|Y_{k_n-1}\|}{C_{k_n-1}}\left(\beta_{k_n-1} - \frac{1}{\mu_{k_n}-\mu_{k_n-1}}\gamma_{k_n}\right),$$
$$\alpha_{k_n} = \frac{\|Y_{k_n}\|}{C_{k_n}}\frac{1}{\mu_{k_n}-\mu_{k_n-1}}\gamma_{k_n}$$

for remaining cases. Then, the norm of state z_0 can be presented as

$$\|z_0\|^2 = \sum_{k\in\mathbb{Z}}|\alpha_k|^2 = \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}\left|\frac{\|Y_k\|}{C_k}\beta_k\right|^2$$
$$+\sum_{k\in\{k_n-1\}}\left|\frac{\|Y_k\|}{C_k}\left(\beta_k - \frac{1}{\mu_{k+1}-\mu_k}\gamma_{k+1}\right)\right|^2$$
$$+\sum_{k\in\{k_n\}}\left|\frac{\|Y_k\|}{C_k}\frac{1}{\mu_k-\mu_{k-1}}\gamma_k\right|^2$$
$$\leq \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}\frac{\|Y_k\|^2}{|C_k|^2}|\beta_k|^2 + \sum_{k\in\{k_n-1\}} 2\frac{\|Y_k\|^2}{|C_k|^2}|\beta_k|^2$$

$$+\sum_{k\in\{k_n\}}\left(2\frac{\|Y_{k-1}\|^2}{|C_{k-1}|^2}+\frac{|Y_k|^2}{|C_k|^2}\right)\frac{1}{|\mu_k-\mu_{k-1}|^2}\|\gamma_k\|^2$$

$$\le\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}c_1^2\frac{|\mu_k^2|^2}{k^2}|\beta_k|^2+\sum_{k\in\{k_n-1\}}c_2^2\frac{|\mu_k^2|^2}{k^2}|\beta_k|^2$$

$$+\sum_{k\in\{k_n\}}c_3^2\left|\mu_k^2\right|^2|\gamma_k|^2, \tag{4.11}$$

where sequences

$$\left(\frac{\frac{\|Y_k\|^2}{|C_k|^2}k^2}{|\mu_k^2|^2}\right)_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}},\quad\left(\frac{2\frac{\|Y_k\|^2}{|C_k|^2}k^2}{|\mu_k^2|^2}\right)_{k\in\{k_n-1\}}$$

and

$$\left(\frac{\frac{1}{|\mu_k-\mu_{k-1}|^2}\left(2\frac{\|Y_{k-1}\|^2}{|C_{k-1}|^2}+\frac{\|Y_k\|^2}{|C_k|^2}\right)}{|\mu_k^2|^2}\right)_{k\in\{k_n\}}$$

are bounded, so there exist constants

$$c_1^2=\sup_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}\frac{\frac{\|Y_k\|^2}{|C_k|^2}k^2}{|\mu_k^2|^2},\ c_2^2=\sup_{k\in\{k_n-1\}}\frac{2\frac{\|Y_k\|^2}{|C_k|^2}k^2}{|\mu_k^2|^2}$$

and

$$c_3^2=\sup_{k\in\{k_n\}}\frac{\frac{1}{|\mu_k-\mu_{k-1}|^2}\left(2\frac{\|Y_{k-1}\|^2}{|C_{k-1}|^2}+\frac{\|Y_k\|^2}{|C_k|^2}\right)}{|\mu_k^2|^2}.$$

Continuing to estimate norm of z_0 of (4.11), we obtain

$$\|z_0\|^2\le\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}c_1^2\left|\mu_k^2\right|^2|\beta_k|^2+\sum_{k\in\{k_n-1\}}c_2^2\left|\mu_k^2\right|^2|\beta_k|^2$$

$$+\sum_{k\in\{k_n\}}c_3^2\left|\mu_k^2\right|^2|\gamma_k|^2$$

$$=\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}c_1^2\left|\mu_k^2\right|^2|\beta_k|^2$$

$$+\sum_{k\in\{k_n-1\}}c_2^2\left|\mu_k^2\beta_k+(\mu_{k+1}+\mu_k)\gamma_{k+1}-(\mu_{k+1}+\mu_k)\gamma_{k+1}\right|^2$$

$$+\sum_{k\in\{k_n\}}c_3^2\left|\mu_k^2\right|^2|\gamma_k|^2$$

$$\le \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} c_1^2 \left|\mu_k^2\right|^2 |\beta_k|^2 + \sum_{k\in\{k_n-1\}} 2c_2^2 \left|\mu_k^2\beta_k + (\mu_{k+1}+\mu_k)\gamma_{k+1}\right|^2$$
$$+ \sum_{k\in\{k_n-1\}} 2c_2^2 |\mu_{k+1}+\mu_k|^2 |\gamma_{k+1}|^2 + \sum_{k\in\{k_n\}} c_3^2 \left|\mu_k^2\right|^2 |\gamma_k|^2$$
$$= \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} c_1^2 \left|\mu_k^2\right|^2 |\beta_k|^2 + \sum_{k\in\{k_n-1\}} 2c_2^2 \left|\mu_k^2\beta_k + (\mu_{k+1}+\mu_k)\gamma_{k+1}\right|^2$$
$$+ \sum_{k\in\{k_n\}} \left(2c_2^2 |\mu_k+\mu_{k-1}|^2 + c_3^2 \left|\mu_k^2\right|^2\right) |\gamma_k|^2$$
$$\le \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} c_1^2 \left|\mu_k^2\right|^2 |\beta_k|^2 + \sum_{k\in\{k_n-1\}} 2c_2^2 \left|\mu_k^2\beta_k + (\mu_{k+1}+\mu_k)\gamma_{k+1}\right|^2$$
$$+ \sum_{k\in\{k_n\}} c_4^2 \left|\mu_k^2\right|^2 |\gamma_k|^2, \tag{4.12}$$

where sequence

$$\left(\frac{2c_2^2 |\mu_k+\mu_{k-1}|^2 + c_3^2 |\mu_k^2|^2}{|\mu_k^2|^2}\right)_{k\in\{k_n\}}$$

is bounded, then there exists constant

$$c_4^2 = \sup_{k\in\{k_n\}} \frac{2c_2^2 |\mu_k+\mu_{k-1}|^2 + c_3^2 |\mu_k^2|^2}{|\mu_k^2|^2}.$$

Then, finally, estimation (4.12) of norm of z_0 takes the form

$$\|z_0\|^2 \le c_5^2 \left(\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} \left|\mu_k^2\right|^2 |\beta_k|^2 \right.$$
$$\left. + \sum_{k\in\{k_n-1\}} \left|\mu_k^2\beta_k + (\mu_{k+1}+\mu_k)\gamma_{k+1}\right|^2 + \sum_{k\in\{k_n\}} \left|\mu_k^2\right|^2 |\gamma_k|^2\right), \tag{4.13}$$

where $c_5^2 = \max\{c_1^2, 2c_2^2, c_4^2\}$. Let $\kappa^2 = \frac{m}{c_5^2}$, combining estimations (4.10) and (4.13), we obtain

$$\kappa^2 \|z_0\|^2 \le m \left(\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} \left|\mu_k^2\right|^2 |\beta_k|^2 + \sum_{k\in\{k_n-1\}} \left|\mu_k^2\beta_k + (\mu_{k+1}+\mu_k)\gamma_{k+1}\right|^2\right.$$
$$\left. + \sum_{k\in\{k_n\}} \left|\mu_k^2\right|^2 |\gamma_k|^2\right)$$
$$\le \|\mathcal{C}\mathcal{T}(t)z_0\|_{H^2}^2,$$

which means that system (4.1) is $H^2(0,T) - \mathcal{H}$ exactly observable for time $T \geq T_0$.

$\square$

Remark 4.8. In Theorem 4.7 we have shown that system (4.1) is $H^2(0,T) - \mathcal{H}$ exactly observable for large times T, that is for $T \geq T_0$, and due to Corollary 4.5 we know that this system cannot be exact observable for small times T, that is for $T < T_0$.

Remark 4.9. Accordingly to Theorem 4.6 the system is not $L^2(0,T) - \mathcal{H}$ observable and according to Theorem 4.7 the system is $H^2(0,T) - \mathcal{H}$ exactly observable. Then the open question arises—what is the optimal smoothness requirement for the left-hand side dividing non-observable and observable systems?

4.3 Observablitity of a Timoshenko Beam

Here we will show the application of exact observability conditions obtained in the previous section for the exact observability problem for Timoshenko beam system governed by (2.12) with boundary condition of the form (2.13).

The deflection of the center line of the beam at the free end will be observed, i.e.

$$Y = \mathcal{C}\begin{pmatrix} w \\ \xi \\ \dot{w} \\ \dot{\xi} \end{pmatrix} = w(1,\cdot). \tag{4.14}$$

Now, we present a few facts about spectral properties of the operator of motion of Timoshenko beam system. Then, we show that the considered system satisfies exact observability conditions stated before, (A1)–(A4).

We start with presenting that condition (A1) is fulfilled. Following [32] we recall some notations: λ_n denotes an increasing sequence of (real, positive) eigenvalues of operator A_1 (see (2.15)), $\sigma_1^{(n)} = \sqrt{\lambda_n - \sqrt{\lambda_n}}$, $\sigma_3^{(n)} = \sqrt{\lambda_n + \sqrt{\lambda_n}}$, $\tau^{(n)} = \tan\frac{\sigma_3^{(n)}}{2}$ if $n \equiv 1, 4 \mod 4$ and $\tau^{(n)} = -\cot\frac{\sigma_3^{(n)}}{2}$ if $n \equiv 2, 3 \mod 4$. The operator $\mathcal{A}$ (see (2.14)) has an orthogonal complete (in $\mathcal{H}$) sequence of eigenelements

$$Y_n = \begin{pmatrix} y_n \\ z_n \\ \mu_n y_n \\ \mu_n z_n \end{pmatrix} \quad \text{and} \quad Y_{-n} = \begin{pmatrix} y_{n+1} \\ z_{n+1} \\ -\mu_{n+1} y_{n+1} \\ -\mu_{n+1} z_{n+1} \end{pmatrix} \quad (n \in \mathbb{N}) \tag{4.15}$$

where

$$y_n(x) = 2\tau^{(n)}\cos\sigma_3^{(n)}x - 2\tau^{(n)}\cos\sigma_1^{(n)}x - 2\sin\sigma_3^{(n)}x \\ - 2\frac{\sigma_1^{(n)}}{\sigma_3^{(n)}}\sin\sigma_1^{(n)}x,$$

$$z_n(x) = 2\tau^{(n)}\frac{\sqrt{\lambda_n}}{\sigma_3^{(n)}}\sin\sigma_3^{(n)}x + 2\tau^{(n)}\frac{\sqrt{\lambda_n}}{\sigma_1^{(n)}}\sin\sigma_1^{(n)}x + 2\frac{\sqrt{\lambda_n}}{\sigma_3}\cos\sigma_3^{(n)}x \\ - 2\frac{\sqrt{\lambda_n}}{\sigma_3^{(n)}}\cos\sigma_1^{(n)}x, \tag{4.16}$$

with corresponding eigenvalues

$$\mu_n = i\sqrt{\lambda_n} = i\begin{cases} \dfrac{2k-1}{2}\pi - \varepsilon_n & \text{if} \quad n = 2k-1 \\ \dfrac{2k-1}{2}\pi + \varepsilon_n & \text{if} \quad n = 2k \end{cases}$$

$$\mu_{-n} = -\mu_{n+1},$$

where $0 < \varepsilon_n$ and $\lim_{n\to\infty}\varepsilon_n = 0$. Proofs of the facts stated above can be found in Lemma 2.2 in [32], Section 3 in [33] and Section 2 in [53].

The following lemmas show that the considered Timoshenko beam system satisfied condition (A2), (A3) and (A4), respectively.

Lemma 4.10. *The difference between a pair of eigenevalues μ_{2k}, μ_{2k-1} is asymptotically equivalent to $\frac{1}{|k|}$, as $|k| \to \infty$, i.e. $|\mu_{2k} - \mu_{2k-1}| \asymp \frac{1}{|k|}$.*

Proof. The proof is a consequence from Lemma on approximation of series in [53]. □

Lemma 4.11 (see Lemma 3.1 in [35])**.** *The system*

$$\left\{e^{\mu_{2k}t}\right\}_{k\in\mathbb{Z}} \cup \left\{\frac{e^{\mu_{2k}t} - e^{\mu_{2k-1}t}}{\mu_{2k} - \mu_{2k-1}}\right\}_{k\in\mathbb{Z}} \tag{4.17}$$

is a Riesz basis for $L^2(0,4)$.

Lemma 4.12. *The norm of the eigenvector Y_k of the operator $\mathcal{A}$ is asymptotically bounded from below and above by k.*

Proof. Let us study the norm of the eigenvector Y_k,

$$\|Y_k\|^2_{\mathcal{H}} = \left\|\begin{pmatrix} y_k \\ z_k \\ \mu_k y_k \\ \mu_k z_k \end{pmatrix}\right\|^2_{\mathcal{H}} = 2|\mu_k|^2\int_0^1 y_k(x)^2 + z_k(x)^2 dx.$$

Using the form of $y_k(x)$ and $z_k(x)$ given in (4.16), we get

$$
\begin{aligned}
\int_0^1 y_k(x)^2 + z_k(x)^2 dx = {} & 2\left(\tau_k^2+1\right)\left(\frac{\lambda}{\sigma_3^2}+1\right) \\
& + \left(\tau_k^2-1\right)\frac{1}{\sigma_3}\left(1-\frac{\lambda}{\sigma_3^2}\right)\sin\left(2\sigma_3\right) \\
& - 4\left(\tau_k^2+\frac{\sigma_1}{\sigma_3}+\frac{\lambda}{\sigma_3^2}+\tau_k^2\frac{\lambda}{\sigma_1\sigma_3}\right)\frac{1}{\sigma_1+\sigma_3}\sin\left(\sigma_1+\sigma_3\right) \\
& + 4\left(\tau_k^2\frac{\lambda}{\sigma_1\sigma_3}-\frac{\lambda}{\sigma_3^2}+\frac{\sigma_1}{\sigma_3}-\tau_k^2\right)\frac{1}{\sigma_1-\sigma_3}\sin\left(\sigma_1-\sigma_3\right) \\
& + 2\tau_k\frac{1}{\sigma_3}\left(1-\frac{\lambda}{\sigma_3^2}\right)\left(\cos\left(2\sigma_3\right)-1\right) \\
& + 2\tau_k\left(\frac{\lambda}{\sigma_1\sigma_3}\frac{1}{\sigma_1}-\frac{1}{\sigma_3}\right) \\
& + 4\tau_k\left(\frac{\lambda}{\sigma_3^2}-\frac{\lambda}{\sigma_1\sigma_3}+\frac{\sigma_1}{\sigma_3}-1\right)\frac{1}{\sigma_1+\sigma_3}\cos\left(\sigma_1+\sigma_3\right) \\
& + 4\tau_k\left(\frac{\sigma_1}{\sigma_3}+1-\frac{\lambda}{\sigma_3^2}-\frac{\lambda}{\sigma_1\sigma_3}\right)\frac{1}{\sigma_1-\sigma_3}\cos\left(\sigma_1-\sigma_3\right) \\
& + 4\tau_k\left(\frac{\lambda}{\sigma_1\sigma_3}-\frac{\sigma_1}{\sigma_3}\right)\frac{2\sigma_1}{\sigma_1^2-\sigma_3^2} \\
& + 4\tau_k\left(\frac{\lambda}{\sigma_3^2}-1\right)\frac{2\sigma_3}{\sigma_1^2-\sigma_3^2} \\
& + \left(\tau_k^2-\frac{\sigma_1^2}{\sigma_3^2}+\frac{\lambda}{\sigma_3^2}-\tau_k^2\frac{\lambda}{\sigma_1^2}\right)\frac{1}{\sigma_1}\sin\left(2\sigma_1\right) \\
& + 2\left(\tau_k^2+\frac{\sigma_1^2}{\sigma_3^2}+\frac{\lambda}{\sigma_3^2}+\tau_k^2\frac{\lambda}{\sigma_1^2}\right).
\end{aligned}
$$

Taking into the fact that [53]

$$
\begin{cases}
\limsup\limits_{k\to\infty}\tau^{(k)} = \dfrac{\tan\frac{1}{4}+1}{1-\tan\frac{1}{4}}, \\
\liminf\limits_{k\to\infty}\tau^{(k)} = \dfrac{\tan\frac{1}{4}-1}{1+\tan\frac{1}{4}},
\end{cases}
$$

we see that $\{\tau_k\}$ is bounded and separated from 0, therefore

$$\begin{cases} \limsup\limits_{|k|\to\infty} \int\limits_0^1 y_k(x)^2 + z_k(x)^2 dx = 8\left(\left(\dfrac{\tan\frac{1}{4}+1}{1-\tan\frac{1}{4}}\right)^2 + 1\right), \\ \liminf\limits_{|k|\to\infty} \int\limits_0^1 y_k(x)^2 + z_k(x)^2 dx = 8\left(\left(\dfrac{\tan\frac{1}{4}-1}{1+\tan\frac{1}{4}}\right)^2 + 1\right). \end{cases}$$

Hence, we obtain, for large k's, that

$$\begin{aligned} \|Y_k\| &\approx \sqrt{16\,|\mu_k|^2\,(\tau_k^2+1)} \\ &\approx M_1\,|\mu_k| \geq M_2|k| \end{aligned}$$

and

$$\|Y_k\| \approx M_1\,|\mu_k| \leq M_3|k|,$$

where M_1, M_2 and M_3 are positive constants. Thus, $\|Y_k\| \asymp |k|$.

$\square$

Similar considerations as in the proof of Lemma 4.12 allow us to state the following

Lemma 4.13. *Observe that*

$$\lim_{|k|\to\infty} |C_k|^2 = \lim_{|k|\to\infty} y_k^2(1) = 16. \tag{4.18}$$

Now Theorems 4.6, 4.7 and Lemmas 4.10–4.13 allow us to state the main result of this chapter.

Theorem 4.14. *The Timoshenko beam system* (2.12)–(2.15) *with observation of a form* (4.14) *is not* $L^2(0,T) - \mathcal{H}$ *exactly observable for any* $T > 0$, *and is* $H^2(0,T) - \mathcal{H}$ *exactly observable for* $T \geq 4$.

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Dissertation summary

autor:	Mateusz Firkowski
title:	Selected problems of stability and observability of Timoshenko beams
supervisor:	Prof. Grigory Sklyar
co-supervisor:	Dr. Jarosław Woźniak
keywords:	Timoshenko beam, asymptotic stability, optimal decay, stability margin, exact observability, Riesz basis, divided differences

The following dissertation is devoted to the analysis of stability and observability of a particular model of vibrations in beams, the so-called Timoshenko beam model.

The structure of the work is as follows: after preface there are four chapters. The first two of them are devoted to the introduction of basic definitions and theorems, which are necessary in the main part of the dissertation.

In the third chapter, we analyze stability of Timoshenko beam model including damping effects. To this end, we carry out spectral analysis of the operators associated with differential equations describing the system under consideration. Then we prove that in some particular cases those operators satisfy spectrum determined growth condition, which means that the location of the spectrum allows us to determine the stability margin of the system. Furthermore, we investigate the existence of an optimal decay rate. At the end we compare the obtained results with other damping operators.

In the fourth chapter, we consider the problem of exact observability of a general class of distributed parameter systems in Hilbert spaces. We prove that the system with some specific assumptions on spectrum and eigensystem is not exactly observable in default topology setting. Then we find stronger topology for state observation for which the system becomes exactly observable. In the end, we show that clamped-free Timoshenko beam system satisfies obtained results.

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Streszczenie rozprawy

autor:	Mateusz Firkowski
tytuł:	Selected problems of stability and observability of Timoshenko beams
promotor:	prof. dr hab. Grigory Sklyar
promotor pomocniczy:	dr Jarosław Woźniak
słowa kluczowe:	belka Tymoszenki, stabilność asymptotyczna, optymalne wygaszanie, zapas stabilności, dokładna obserwowalność, baza Riesza, ilorazy różnicowe

Niniejsza rozprawa poświęcona jest analizie stabilności i obserwowalności szczególnego modelu drgań występujących w belkach, tak zwanemu modelowi belki Timoszenki.

Struktura pracy jest następująca: po przedmiowie znajdują się cztery rozdziały. Dwa pierwsze z nich poświęcone są wprowadzeniu podstawowych twierdzeń i definicji, które są niezbędne w głównej części rozprawy.

W trzecim rozdziale analizujemy stabilność modelu belki Timoszenki z uwzględnieniem efektów tłumienia. W tym celu przeprowadzona została analiza spektralna operatorów związanych z równaniami różniczkowymi opisującymi rozważany układ. Następnie udowadniamy, że w niektórych przypadkach operatory te spełniają spektralny warunek wzrostu, co oznacza, że położenie spektrum pozwala nam wyznaczyć zapas stabilności układu. Ponadto, badamy istnienie optymalnego współczynnika wygaszania. Na koniec porównujemy uzyskane wyniki z innymi operatorami wygaszania.

W czwartym rozdziale rozważamy problem dokładnej obserwowalności ogólnej klasy układów z rozproszonymi parametrami w przestrzeniach Hilberta. Udowodniliśmy, że układ z pewnymi szczególnymi założeniami dotyczącymi spektrum i układu własnego, nie jest dokładnie obserwowalny w domyślnej topologii. Następnie znajdujemy silniejszą topologię dla obserwacji stanu, dla której układ staje się dokładnie obserwowalny. Pokazujemy, że zaczepiona belka Timoshenki spełnia otrzymane założenia.

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